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Theory of Microcubes

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Kock and Lavendhomme have begun to couch the standard theory of iterated tangents within the due framework of synthetic differential geometry. Generalizing their theory of microsquares, we give a general theory of microcubes, its three-dimensional generalization, in which an unexpected generalization of the Jacobi identity of vector fields with respect to Lie brackets and a synthetic treatment of Bianchi's first identity are discussed.

INTRODUCTION

In order to get a unified theory of physics in which relativity and quantum theory are concordant with each other, we must quantize geometry or the pure science of space. To this end, we must choose the right geometry to be quantized, and we believe that it is not standard differential geometry and the category of smooth manifolds, but synthetic differential geometry and the category of microlinear spaces that are truly susceptible of quantization. For textbooks on synthetic differential geometry and microlinear space in particular, the reader is referred to Lavendhomme (1996) and Moerdijk and Reyes (1991). For the first attempt to quantize synthetic differential geometry, the reader is referred to Nishimura (1996a,b). Some treatments of Hamiltonian and Lagrangian mechanics within the framework of synthetic differential geometry can be seen in Nishimura (1997, n.d.-a,b).

The theory of iterated tangents such as seen in White (1982) and Yano and Ishihara (1973), which deals somewhat clumsily with higher order structures of infinitesimals without explicitly referring to them, should be regarded as a precursor of synthetic differential geometry. Its basic ideas still remain to be couched within the framework of synthetic differential geometry, in which various kinds of infinitesimals are generously available. The first

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attempt in this direction was tried by Kock and Lavendhomme (1984), who discussed strong differences of microsquares and applied them to Lie brackets of vector fields and affine connections. This paper develops three-dimensional generalizations of their two-dimensional ideas, paving the way to still higher dimensional generalizations. The paper consists of six sections, the first three culminating in an interesting generalization of the Jacobi identity of vector fields with respect to Lie brackets (Theorem 3.1), while the last three are organized toward a synthetic treatment of Bianchi's first identity (Theorem 6.4).

Let M be a microlinear space with $m \in M$. These entities shall be fixed throughout this paper. We assume the reader is well familiar with Lavendhomme's (1996) readable textbook on synthetic differential geometry up to Chapter 5, though we could not resist the temptation to use some of our own notation and terminology. As is usual in synthetic differential geometry, the reader should presume that we are working in a topos, so that the principle of excluded middle and Zorn's lemma should be avoided. But for these two points, we could feel that we are working in the standard universe of sets.

It is well known in synthetic differential geometry that the set $\mathrm{T}^1(M; m)$ of tangent vectors at m is an R-module, where R is intended as the set of real numbers containing plenty of infinitesimals and abiding by the so-called general Kock axiom. Given $t_1, \ldots, t_n \in \mathrm{T}^1(M; m)$, there exists a unique function $l_{(t_1,\ldots,t_n)}\colon D(n) \to M$ such that

$$l_{(t_1,\ldots,t_n)} \circ \iota_i \qquad (i = 1, \ldots, n)$$

where ι_i is the ith canonical injection of D into $D(n)$. The sum $t_1 + \cdots + t_n$ is then given as follows:

(0.1) $(t_1 + \cdots + t_n)(d) = l_{(t_1,\ldots,t_n)}(d, \ldots, d)$ for any $d \in D$.

Kock and Lavendhomme (1984) studied the set $\mathrm{T}^2(M; m)$ of microsquares at m by introducing the notion of strong difference $\dot{-}$. Given $\alpha_1, \alpha_2 \in \mathrm{T}^2(M; m)$, their strong difference $\alpha_2 \dot{-} \alpha_1$ is defined, so long as they coincide on the axes, in which $\alpha_2 \dot{-} \alpha_1$ lies in $\mathrm{T}^1(M; m)$. Identifying the set $\chi^1(M)$ of vector fields on M with the tangent space $\mathrm{T}^1(M^M; \mathrm{id}_M)$ of M^M at the identity transformation id_M of M, they could express the Lie bracket $[X, Y]$ of $X, Y \in \chi^1(M)$ in terms of strong difference in $\mathrm{T}^2(M^M; \mathrm{id}_M)$. However, they stopped at this point without noticing that this may lead to a new synthetic proof of the Jacobi identity of vector fields. Section 1 is devoted to a brief review on some of their story.

Our story begins exactly where they stopped. We remark that, just as the space $\mathrm{T}^2(M^M; \mathrm{id}_M)$ provided a good framework for discussing the Lie bracket $[X, Y]$ of vector fields X and Y on M, the space $\mathrm{T}^3(M^M; \mathrm{id}_M)$ of microcubes on M^M at id_M provides a good framework for discussing combinations of three vector fields X, Y, and Z on M by Lie brackets such as $[X, [Y, Z]]$. The so-called Jacobi identity claims precisely that the sum of the three vector fields obtained from $[X, [Y, Z]]$ by cyclically permuting among X, Y, and Z vanishes. In Section 2 we study the space $\mathsf{T}^3(M; m)$ of microcubes on M at m by introducing three kinds of strong difference, to be denoted by $\underset{1}{\div}$, $\underset{2}{\div}$, and $\underset{3}{\div}$, which correspond to three different ways of viewing microcubes on M as microsquares on M^D. The interactions of the three strong differences make the theory of microcubes more than a prosaic variant of the theory of microsquares, just as the interactions of addition and multiplication make the theory of rings something more than the theory of groups or the theory of monoids. Such combinations of three vector fields X, Y, and Z on M by Lie brackets as $[X, [Y, Z]]$ are shown to be expressible in terms of the three strong differences $\underset{1}{\div}$, $\underset{2}{\div}$, and $\underset{3}{\div}$ in $\mathsf{T}^3(M^M; \mathrm{id}_M)$ and the strong difference $\dot{-}$ in $\mathsf{T}^2(M^M; \mathrm{id}_M)$.

Section 3 is devoted to the general Jacobi identity in $\mathsf{T}^3(M; m)$, which is expressed in terms of the three strong differences $\underset{1}{\div}$, $\underset{2}{\div}$, and $\underset{3}{\div}$ in $\mathsf{T}^3(M; m)$ and the strong difference $\dot{-}$ in $\mathsf{T}^2(M; m)$. Since all four strong differences are partial operations, it is not always meaningful. However, whenever it is meaningful, it is logically forced to obtain. The classical Jacobi identity of vector fields is only an easy corollary of this far-reaching Jacobi identity for microcubes, or using a locution from orthodox Marxism, the former is the superstructure of the latter. It is our great surprise that the Jacobi identity should hold at such a deeper level. The proof of the general Jacobi identity requires construction of a quasi-colimit diagram of small objects, which is too gigantic for us to pass through in saying simply that, in order to see that it is indeed a quasi-colimit diagram of small objects, it suffices to try it in the case of $M = \mathbf{R}$. Therefore we investigate the general Jacobi identity in full detail in the simple case of $M = \mathbf{R}$ before treating it generally, though we know well that the case of $M = \mathbf{R}$ requires no more than high school mathematics by dint of the general Kock axiom. We have chosen this route, because we would like to make the paper more than a fossil in the Bourbaki style. We believe that this choice will enable the reader to share our aspiration and excitement.

Section 5 deals with another strong difference of microcubes. This section could be read just after Section 2, and Theorem 5.4 and its proof might prepare the reader for more advanced Theorem 3.1 and its proof. Section 6 can be regarded as a solution of White's (1982) Exercise 19 of Chapter 4 from a synthetic viewpoint. The proof, without making use of the Jacobi identity of vector fields, deepens our geometric comprehension of the first Bianchi identity.

Section 4 deals with a subject which is more general than the title of the paper suggests. It is concerned with a distinguishable class of small objects called simplicial objects. Since the main topic of the paper is not simplicial objects in general but microcubes, our exposition is kept to the point that is requisite for later sections. Therefore it is superficial, far from exhaustive. In Sections 4 and 6 a connection ∇ in the sense of Lavendhomme (1996, §5.1, Definition 1) is supposed to exist on M.

The rest of this section is devoted to miscellaneous remarks on notation and terminology.

(1) Such standard notations for small objects as 1, D, $D(2)$, and $D^2 = D \times D$ are used freely.

(2) Given a small object E, the unique morphism $1 \to E$ is generally denoted 0.

(3) A diagram $\{E_\lambda \xrightarrow{\eta_\lambda} F\}$ of small objects is called a *quasi-colimit diagram of small objects* if

$$\{\mathbb{R}^F \xrightarrow{\mathbb{R}^{\eta_\lambda}} \mathbb{R}^{E_\lambda}\}$$

is a limit diagram, in which F is called the *quasi-colimit* of the diagram.

(4) If there is a canonical injection of a small object E into a small object F, it is generally denoted by ι.

(5) The totality of n-microcubes on M at m is denoted by $\mathrm{T}^n(M; m)$.

(6) The set $\mathrm{T}^n(M^M; \mathrm{id}_M)$ is denoted by $\chi^n(M)$. In particular, $\chi^1(M)$ is the set of vector fields on M.

(7) Given $X^1, \ldots, X^n \in \chi^1(M)$, we denote by $X^n * \cdots * X^1$ the element of $\chi^n(M)$ such that

$$\text{(0.2)} \quad (X^n * \cdots * X^1)(d_1, \ldots, d_n) = X^n_{d_n} \circ \cdots \circ X^1_{d_1}$$

for any $(d_1, \ldots, d_n) \in D^n$.

(8) The group of permutations of the set $\{1, \ldots, n\}$ is denoted by $\mathfrak{Perm}_n$. Cycles are denoted (12), (123), etc. For example, (123) denotes the permutation of the set $\{1, \ldots, n\}$ assigning 2 to 1, 3 to 2, and 1 to 3 while keeping the other elements fixed. Cycles of length 2 are called transpositions. It is well known that the group $\mathfrak{Perm}_n$ is generated by transpositions.

(9) If $\{D_\lambda \xrightarrow{f_{\mu\lambda}} D_\mu\}$ is a diagram of small objects, then the diagram *perceived by* $\mathbb{R}$ means the diagram

$$\{\mathbb{R}^{D_\mu} \xrightarrow{\mathbb{R}^{f_{\mu\lambda}}} \mathbb{R}^{D_\lambda}\}$$

This locution will appear in the proofs of Lemmas 3.3 and 5.5.

1. MICROSQUARES

This section is completely a review, and the reader is referred to Lavendhomme (1996, §§3.4) for details.

Lemma 1.1. The diagram

$$\begin{tikzcd} D(2) \arrow[r, "i"] \arrow[d, "i"'] & D^2 \arrow[d, "\psi"] \\ D^2 \arrow[r, "\varphi"'] & D^2 \vee D \end{tikzcd}$$

is a quasi-colimit diagram of small objects, where

(1.1) $D^2 \vee D = \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = d_2 d_3 = 0\}$.
(1.2) $\varphi(d_1, d_2) = (d_1, d_2, 0)$ for any $(d_1, d_2) \in D^2$.
(1.3) $\psi(d_1, d_2) = (d_1, d_2, d_1 d_2)$ for any $(d_1, d_2) \in D^2$.

The above lemma enables us to define the notion of strong difference for microsquares as follows:

Proposition 1.2. For any $\alpha_1, \alpha_2 \in \mathbf{T}^2(M; m)$, if $\alpha_1|_{D(2)} = \alpha_2|_{D(2)}$, then there exists a unique function $\mathscr{g}_{(\alpha_1,\alpha_2)}: D^2 \vee D \to M$ such that $\mathscr{g}_{(\alpha_1,\alpha_2)} \circ \varphi = \alpha_1$ and $\mathscr{g}_{(\alpha_1,\alpha_2)} \circ \psi = \alpha_2$. In this case we define a tangent vector $\alpha_2 \dot{-} \alpha_1$ in $\mathbf{T}^1(M; m)$ as follows:

(1.4) $(\alpha_2 \dot{-} \alpha_1)(d) = \mathscr{g}_{(\alpha_1,\alpha_2)}(0, 0, d)$ for any $d \in D$.

Lemma 1.3. The diagram

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "0"'] & D \arrow[d, "\epsilon"] \\ D^2 \arrow[r, "\varphi"'] & D^2 \vee D \end{tikzcd}$$

is a quasi-colimit diagram of small objects, where

(1.5) $\epsilon(d) = (0, 0, d)$ for any $d \in D$.

The above lemma enables us to define the notion of strong translation for microsquares as follows:

Proposition 1.4. For any $t \in \mathrm{T}^1(M; m)$ and any $\alpha \in \mathrm{T}^2(M; m)$, there exists a unique function ℓ: $D^2 \vee D \to M$ such that $\ell_{(t,\alpha)} \circ \epsilon = t$ and $\ell_{(t,\alpha)} \circ \varphi = \alpha$. In this case we define a microsquare $t \dotplus \alpha$ in $\mathrm{T}^2(M; m)$ as follows:

(1.6) $(t \dotplus \alpha)(d_1, d_2) = \ell_{(t,\alpha)}(d_1, d_2, d_1 d_2)$ for any $(d_1, d_2) \in D^2$.

Given $\alpha \in \mathrm{T}^2(M; m)$, we define $\Sigma(\alpha) \in \mathrm{T}^2(M; m)$ as follows:

(1.7) $\Sigma(\alpha)(d_1, d_2) = \alpha(d_2, d_1)$ for any $(d_1, d_2) \in D^2$.

Proposition 1.5. For any $\alpha_1, \alpha_2 \in \mathrm{T}^2(M; m)$, if $\alpha_1|_{D(2)} = \alpha_2|_{D(2)}$, then

(1.8) $\Sigma(\alpha_2) \dot{-} \Sigma(\alpha_1) = \alpha_2 \dot{-} \alpha_1$

The following result has inspired our study of the Jacobi identity within the context of microcubes.

Proposition 1.6. For any $X, Y \in \chi^1(M)$, the right-hand side of the following equality is meaningful and the equality holds:

(1.9) $[X, Y] = Y * X \dot{-} \Sigma(X * Y)$

2. STRONG DIFFERENCES IN MICROCUBES

The following lemma is a three-dimensional generalization of Lemma 1.1.

Lemma 2.1. The diagrams

$$\begin{tikzcd} D[3; 2, 3] \arrow[r, "\iota"] \arrow[d, "\iota"'] & D^3 \arrow[d, "\psi_1"] \\ D^3 \arrow[r, "\varphi_1"'] & D^3 \underset{1}{\vee} D \end{tikzcd}$$

$$\begin{tikzcd} D[3; 1, 3] \arrow[r, "\iota"] \arrow[d, "\iota"'] & D^3 \arrow[d, "\psi_2"] \\ D^3 \arrow[r, "\varphi_2"'] & D^3 \underset{2}{\vee} D \end{tikzcd}$$

$$\begin{tikzcd} D[3;\,1,2] \arrow[r, "\iota"] \arrow[d, "\iota"'] & D_3 \arrow[d, "\psi_3"] \\ D^3 \arrow[r, "\varphi_3"'] & D^3 \underset{3}{\vee} D \end{tikzcd}$$

are quasi-colimit diagrams of small objects, where

(2.1) $D[3;\,1,2] = D(2) \times D = \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\}$
(2.2) $D[3;\,2,3] = D \times D(2) = \{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\}$
(2.3) $D[3;\,1,3] = \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\}$
(2.4) $D^3 \underset{1}{\vee} D = \{(d_1, d_2, d_3, d_4) \in D^4 \mid d_2 d_4 = d_3 d_4 = 0\}$
(2.5) $D^3 \underset{2}{\vee} D = \{(d_1, d_2, d_3, d_4) \in D^4 \mid d_1 d_4 = d_3 d_4 = 0\}$
(2.6) $D^3 \underset{3}{\vee} D = \{(d_1, d_2, d_3, d_4) \in D^4 \mid d_1 d_4 = d_2 d_4 = 0\}$
(2.7) $\varphi_1(d_2, d_2, d_3) = \varphi_2(d_1, d_2, d_3) = \varphi_3(d_1, d_2, d_3) = (d_2, d_2, d_3, 0)$ for any $(d_1, d_2, d_3) \in D^3$
(2.8) $\psi_1(d_1, d_2, d_3) = (d_1, d_2, d_3, d_2 d_3)$ for any $(d_1, d_2, d_3) \in D^3$
(2.9) $\psi_2(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1 d_3)$ for any $(d_1, d_2, d_3) \in D^3$
(2.10) $\psi_3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1 d_2)$ for any $(d_1, d_2, d_3) \in D^3$

As we will see below, we have three kinds of strong difference for microcubes.

Proposition 2.2. (1) For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M; m)$, if $\gamma_1|_{D[3;2,3]} = \gamma_2|_{D[3;2,3]}$, then there exists a unique function $\wp^1_{(\gamma_1,\gamma_2)}: D^3 \underset{1}{\vee} D \to M$ such that $\wp^1_{(\gamma_1,\gamma_2)} \circ \varphi_1 = \gamma_1$ and $\wp^1_{(\gamma_1,\gamma_2)} \circ \psi_1 = \gamma_2$. In this case we define a microsquare $\gamma_2 \underset{1}{\div} \gamma_1$ in $\mathsf{T}^2(M; m)$ as follows:

(2.11) $(\gamma_2 \underset{1}{\div} \gamma_1)(d_1, d_2) = \wp^1_{(\gamma_1,\gamma_2)}(d_1, 0, 0, d_2)$ for any $(d_1, d_2) \in D^2$.

(2) For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M; m)$, if $\gamma_1|_{D[3;1,3]} = \gamma_2|_{D[1;2,3]}$, then there exists a unique function $\wp^2_{(\gamma_1,\gamma_2)}: D^3 \underset{2}{\vee} D \to M$ such that $\wp^2_{(\gamma_1,\gamma_2)} \circ \varphi_2 = \gamma_1$ and $\wp^2_{(\gamma_1,\gamma_2)} \circ \psi_2 = \gamma_2$. In this case we define a microsquare $\gamma_2 \underset{2}{\div} \gamma_1$ in $\mathsf{T}^2(M; m)$ as follows:

(2.12) $(\gamma_2 \underset{2}{\div} \gamma_1)(d_1, d_2) = \wp^2_{(\gamma_1,\gamma_2)}(0, d_1, 0, d_2)$ for any $(d_1, d_2) \in D^2$.

(3) For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M; m)$, if $\gamma_1|_{D[3;1,2]} = \gamma_2|_{D[1;1,2]}$, then there exists a unique function $\wp^3_{(\gamma_1,\gamma_2)}: D^2 \underset{3}{\vee} D \to M$ such that $\wp^3_{(\gamma_1,\gamma_2)} \circ \varphi_3 = \gamma_1$ and $\wp^3_{(\gamma_1,\gamma_2)} \circ \psi_3 = \gamma_2$. In this case we define a microsquare $\gamma_2 \underset{3}{\div} \gamma_1$ in $\mathsf{T}^2(M; m)$ as follows:

(2.13) $(\gamma_2 \underset{3}{\div} \gamma_1)(d_1, d_2) = \wp^3_{(\gamma_1,\gamma_2)}(0, 0, d_1, d_2)$ for any $(d_1, d_2) \in D^2$.

Proof. Follows directly from Lemma 2.1. ∎

It is not difficult to see that the three kinds of strong difference in the above proposition correspond to three ways of viewing microcubes on M as microsquares on M^D. What makes the theory of microcubes more than a prosaic three-dimensional generalization of Kock and Lavendhomme's (1984) theory of strong difference for microsquares is the interaction among the three kinds of strong difference.

We do not need the notion of strong translation in microcubes as far as our principal application (i.e., the Jacobi identity) is concerned, but we will present its definition for completeness.

Lemma 2.3. The diagrams

$$\begin{tikzcd} D \arrow[r, "\iota_1^2"] \arrow[d, "\iota_3^3"'] & D^2 \arrow[d, "\epsilon_1"] \\ D^3 \arrow[r, "\varphi_1"'] & D^3 \underset{1}{\vee} D \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, "\iota_1^2"] \arrow[d, "\iota_2^3"'] & D^2 \arrow[d, "\epsilon_2"] \\ D^3 \arrow[r, "\varphi_2"'] & D^3 \underset{2}{\vee} D \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, "\iota_1^2"] \arrow[d, "\iota_3^3"'] & D^2 \arrow[d, "\epsilon_3"] \\ D^3 \arrow[r, "\varphi_3"'] & D^3 \underset{3}{\vee} D \end{tikzcd}$$

are quasi-colimit diagrams of small objects, where

(2.14) $\iota_1^2(d) = (d, 0)$ for any $d \in D$.
(2.15) $\iota_1^3(d) = (d, 0, 0)$ for any $d \in D$.
(2.16) $\iota_2^3(d) = (0, d, 0)$ for any $d \in D$.
(2.17) $\iota_3^3(d) = (0, 0, d)$ for any $d \in D$.
(2.18) $\epsilon_1(d_1, d_2) = (d_1, 0, 0, d_2)$ for any $(d_1, d_2) \in D^2$.
(2.19) $\epsilon_2(d_1, d_2) = (0, d_1, 0, d_2)$ for any $(d_1, d_2) \in D^2$.
(2.20) $\epsilon_3(d_1, d_2) = (0, 0, d_1, d_2)$ for any $(d_1, d_2) \in D^2$.

Proposition 2.4. (1) For any $\alpha \in \mathsf{T}^2(M; m)$ and any $\gamma \in \mathsf{T}^3(M; m)$, if $\alpha \circ \iota_1^2 = \gamma \circ \iota_1^3$, then there exists a unique function $\ell^1_{(\alpha,\gamma)} \circ \epsilon_1 = \alpha$ and $\ell^1_{(\alpha,\gamma)} \circ \varphi_1 = \gamma$. In this case we define a microcube $\alpha \underset{1}{\dotplus} \gamma$ in $\mathsf{T}^3(M; m)$ as follows:

(2.21) $(\alpha \underset{1}{\dotplus} \gamma)(d_1, d_2, d_3) = \ell^1_{(\alpha,\gamma)}(d_1, d_2, d_3, d_2d_3)$ for any $(d_1, d_2, d_3) \in D^3$.

(2) For any $\alpha \in \mathsf{T}^2(M; m)$ and any $\gamma \in \mathsf{T}^3(M; m)$, if $\alpha \circ \iota_1^2 = \gamma \circ \iota_2^3$, then there exists a unique function $\ell^2_{(\alpha,\gamma)} \circ \epsilon_2 = \alpha$ and $\ell^2_{(\alpha,\gamma)} \circ \varphi_2 = \gamma$. In this case we define a microcube $\alpha \underset{2}{\dotplus} \gamma$ in $\mathsf{T}^3(M; m)$ as follows:

(2.22) $(\alpha \underset{2}{\dotplus} \gamma)(d_1, d_2, d_3) = \ell^2_{(\alpha,\gamma)}(d_1, d_2, d_3, d_1d_3)$ for any $(d_1, d_2, d_3) \in D^3$.

(3) For any $\alpha \in \mathsf{T}^2(M; m)$ and any $\gamma \in \mathsf{T}^3(M; m)$, if $\alpha \circ \iota_1^2 = \gamma \circ \iota_3^3$, then there exists a unique function $\ell^3_{(\alpha,\gamma)} \circ \epsilon_3 = \alpha$ and $\ell^3_{(\alpha,\gamma)} \subseteq \varphi_3 = \gamma$. In this case we define a microcube $\alpha \underset{3}{\dotplus} \gamma$ in $\mathsf{T}^3(M; m)$ as follows:

(2.23) $(\alpha \underset{3}{\dotplus} \gamma)(d_1, d_2, d_3) = \ell^3_{(\alpha,\gamma)}(d_1, d_2, d_3, d_1d_2)$ for any $(d_1, d_2, d_3) \in D^3$.

Proof. Follows directly from Lemma 2.3. ∎

It is not difficult to see that the three kinds of strong translation in the above proposition correspond to three ways of viewing microcubes on M as microsquares on M^D.

Leaving a prosy modification of Kock and Lavendhomme's (1984) theory of strong difference for each of the three kinds of strong difference for microcubes to the reader, we now turn to their interactions and their applications to Lie brackets of vector fields.

Given $\gamma \in \mathsf{T}^3(M; m)$ and $\rho \in \mathfrak{Perm}_3$, the microcube $\Sigma_\rho(\gamma)$ in $\mathsf{T}^3(M; m)$ is defined as follows:

(2.24) $\Sigma_\rho(\gamma)(d_1, d_2, d_3) = \gamma(d_{\rho(1)}, d_{\rho(2)}, d_{\rho(3)})$ for any $(d_1, d_2, d_3) \in D^3$.

Permutations in $\mathfrak{Perm}_3$ induce permutations among the three strong differences $\underset{1}{\div}$, $\underset{2}{\div}$, and $\underset{3}{\div}$ for microcubes. Since every permutation can be written as a composition of transpositions, it suffices to see how transpositions such as (12) affect the three strong differences. The following proposition should be obvious.

Proposition 2.5. (1) For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M; m)$, $\gamma_1|_{D[3;2,3]} = \gamma_2|_{D[3;2,3]}$ iff

$$\Sigma_{(23)}(\gamma_1)|_{D[3;2,3]} = \Sigma_{(23)}(\gamma_2)|_{D[3;2,3]}$$

If any of the two equivalent conditions (therefore both of them) holds, then

(2.25) $\gamma_2 \underset{1}{\div} \gamma_1 = \Sigma_{(23)}(\gamma_2) \underset{1}{\div} \Sigma_{(23)}(\gamma_1)$.

(2) For any $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$, $\gamma_1|_{D[3;1,3]} = \gamma_2|_{D[3;1,3]}$ iff

$$\Sigma_{(13)}(\gamma_1)|_{D[3;1,3]} = \Sigma_{(13)}(\gamma_2)|_{D[3;1,3]}$$

If any of the two equivalent conditions (therefore both of them) holds, then

$$(2.26) \quad \gamma_2 \underset{2}{\dot{-}} \gamma_1 = \Sigma_{(13)}(\gamma_2) \underset{2}{\dot{-}} \Sigma_{(13)}(\gamma_1).$$

(3) For any $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$, $\gamma_1|_{D[3;1,2]} = \gamma_2|_{D[3;1,2]}$ iff

$$\Sigma_{(12)}(\gamma_1)|_{D[3;1,2]} = \Sigma_{(12)}(\gamma_2)|_{D[3;1,2]}$$

If any of the two equivalent conditions (therefore both of them) holds, then

$$(2.27) \quad \gamma_2 \underset{3}{\dot{-}} \gamma_1 = \Sigma_{(12)}(\gamma_2) \underset{3}{\dot{-}} \Sigma_{(12)}(\gamma_1).$$

(4) For any $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$, if $\gamma_1|_{D[3;2,3]} = \gamma_2|_{D[3;2,3]}$, then

$$\Sigma_{(12)}(\gamma_1)|_{D[3;1,3]} = \Sigma_{(12)}(\gamma_2)|_{D[3;1,3]}$$

and

$$\Sigma_{(13)}(\gamma_1)|_{D[3;1,2]} = \Sigma_{(13)}(\gamma_2)|_{D[3;1,2]}$$

In this case we have

$$(2.28) \quad \gamma_2 \underset{1}{\dot{-}} \gamma_1 = \Sigma_{(12)}(\gamma_2) \underset{2}{\dot{-}} \Sigma_{(12)}(\gamma_1) = \Sigma_{(13)}(\gamma_2) \underset{3}{\dot{-}} \Sigma_{(13)}(\gamma_1)$$

(5) For any $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$, if $\gamma_1|_{D[3;1,3]} = \gamma_2|_{D[3;1,3]}$, then

$$\Sigma_{(12)}(\gamma_1)|_{D[3;2,3]} = \Sigma_{(12)}(\gamma_2)|_{D[3;2,3]}$$

and

$$\Sigma_{(23)}(\gamma_1)|_{D[3;1,2]} = \Sigma_{(23)}(\gamma_2)|_{D[3;1,2]}$$

In this case we have

$$(2.29) \quad \gamma_2 \underset{2}{\dot{-}} \gamma_1 = \Sigma_{(12)}(\gamma_2) \underset{1}{\dot{-}} \Sigma_{(12)}(\gamma_1) = \Sigma_{(23)}(\gamma_2) \underset{3}{\dot{-}} \Sigma_{(23)}(\gamma_1)$$

(6) For any $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$, if $\gamma_1|_{D[3;1,2]} = \gamma_2|_{D[3;1,2]}$, then

$$\Sigma_{(13)}(\gamma_1)|_{D[3;2,3]} = \Sigma_{(13)}(\gamma_2)|_{D[3;2,3]}$$

and

$$\Sigma_{(23)}(\gamma_1)|_{D[3;1,3]} = \Sigma_{(23)}(\gamma_2)|_{D[3;1,3]}$$

In this case we have

$$(2.30) \quad \gamma_2 \underset{3}{\dot{-}} \gamma_1 = \Sigma_{(13)}(\gamma_2) \underset{1}{\dot{-}} \Sigma_{(13)}(\gamma_1) = \Sigma_{(23)}(\gamma_2) \underset{2}{\dot{-}} \Sigma_{(23)}(\gamma_1)$$

We close this section by relating our three-dimensional theory of strong differences to Lie brackets of vector fields.

Given $X \in \chi^1(M)$ and $U \in \chi^2(M)$, we define $X * U$ and $U * X$ in $\chi^3(M)$ as follows:

(2.31) $(U * X)(d_1, d_2, d_3) = U(d_2, d_3) \circ X(d_1)$ for any $(d_1, d_2, d_3) \in D^3$.
(2.32) $(X * U)(d_1, d_2, d_3) = X(d_3) \circ U(d_1, d_2)$ for any $(d_1, d_2, d_3) \in D^3$.

The following proposition should be obvious.

Proposition 2.6. For any $X \in \chi^1(M)$ and any $U, V \in \chi^2(M)$ with $U|D(2) = V|D(2)$, we have the following:

(2.33) $U * X|_{D[3;2,3]} = V * X|_{D[3;2,3]}$
(2.34) $X * U|_{D[3;1,2]} = X * V|_{D[3;1,2]}$
(2.35) $U * X \underset{1}{\dot{-}} V * X = (U \dot{-} V) * X$
(2.36) $X * U \underset{3}{\dot{-}} X * V = \Sigma(X * (U \dot{-} V))$

The following simple proposition suggests how our burgeoning theory of microcubes is intimately related with the so-called Jacobi identity of vector fields. This point will be deepened in the next section.

Proposition 2.7. Given $X, Y, Z \in \chi^1(M)$, let it be the case that

(2.37) $\gamma_{123} = Z * Y * X$
(2.38) $\gamma_{132} = \Sigma_{(23)}(Y * Z * X)$
(2.39) $\gamma_{213} = \Sigma_{(12)}(Z * X * Y)$
(2.40) $\gamma_{231} = \Sigma_{(123)}(X * Z * Y)$
(2.41) $\gamma_{312} = \Sigma_{(132)}(Y * X * Z)$
(2.42) $\gamma_{321} = \Sigma_{(13)}(X * Y * Z)$

Then the right-hand sides of the following three identities are meaningful, and all the three identities hold:

(2.43) $[X, [Y, Z]] = (\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$
(2.44) $[Y, [Z, X]] = (\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})$
(2.45) $[Z, [X, Y]] = (\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})$

Proof. Follows from Propositions 1.5, 1.6, 2.5, and 2.6. ∎

3. THE GENERAL JACOBI IDENTITY

Entirely distinct from in standard differential geometry, the so-called Jacobi identity of vector fields with respect to Lie brackets occupies a naggingly ticklish position in synthetic differential geometry. Several synthetic proofs of it are known, namely, Kock (1981), Lavendhomme (1996), Nishimura (1997), and Reyes and Wraith (1978), among which Nishimura (1997) is the most elegant. This section is by no means intended to give another nice proof of it, but to probe into a deeper structure making it effective. The main result of the section is the following unexpected generalization of the identity, whose proof will show how and why the Jacobi identity should hold with respect to Lie brackets of vector fields.

Theorem 3.1. Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in T^3(M; m)$. As far as all of the following three expressions are well defined, they sum up only to vanish:

(3.1) $(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$
(3.2) $(\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})$
(3.3) $(\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})$

Before embarking upon a proof of the above theorem, we note that the celebrated Jacobi identity of vector fields with respect to Lie brackets is a direct consequence of it. White's (1982, p. 100) Exercise 11 of Chapter 2 dealt with the Jacobi identity of vector fields on manifolds by using his favorite method of iterated tangents.

Theorem 3.2. For any $X, Y, Z \in \chi^1(M)$, we have the following Jacobi identity:

(3.4) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Proof. Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321}$ be as in Proposition 2.7. Then (3.4) follows from Proposition 2.7 and the above theorem. ■

To understand how Theorem 3.1 prevails, let us ponder the simple case that $M = \mathbf{R}$ and $m = 0$ (the latter condition is inessential, but it is taken for the sake of simplicity), in which the so-called general Kock axiom (cf. Lavendhomme, 1996, §2.1.3) warrants that the microcubes $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}$, and γ_{321} at issue are polynomials of d_1, d_2, and d_3 in D with coefficients in R:

(3.5) $\gamma_{123}(d_1, d_2, d_3) = a_x^{123}d_1 + a_y^{123}d_2 + a_z^{123}d_3 + a_{xy}^{123}d_1d_2 + a_{yz}^{123}d_2d_3 + a_{xz}^{123}d_1d_3 + a_{xyz}^{123}d_1d_2d_3$
(3.6) $\gamma_{132}(d_1, d_2, d_3) = a_x^{132}d_1 + a_y^{132}d_2 + a_z^{132}d_3 + a_{xy}^{132}d_1d_2 + a_{yz}^{132}d_2d_3 + a_{xz}^{132}d_1d_3 + a_{xyz}^{132}d_1d_2d_3$
(3.7) $\gamma_{213}(d_1, d_2, d_3) = a_x^{213}d_1 + a_y^{213}d_2 + a_z^{213}d_3 + a_{xy}^{213}d_1d_2 + a_{yz}^{213}d_2d_3 + a_{xz}^{213}d_1d_3 + a_{xyz}^{213}d_1d_2d_3$
(3.8) $\gamma_{231}(d_1, d_2, d_3) = a_x^{231}d_1 + a_y^{231}d_2 + a_z^{231}d_3 + a_{xy}^{231}d_1d_2 + a_{yz}^{231}d_2d_3 + a_{xz}^{231}d_1d_3 + a_{xyz}^{231}d_1d_2d_3$
(3.9) $\gamma_{312}(d_1, d_2, d_3) = a_x^{312}d_1 + a_y^{312}d_2 + a_z^{312}d_3 + a_{xy}^{312}d_1d_2 + a_{yz}^{312}d_2d_3 + a_{xz}^{312}d_1d_3 + a_{xyz}^{312}d_1d_2d_3$

(3.10) $\gamma_{321}(d_1, d_2, d_3) = a_x^{321}d_1 + a_y^{321}d_2 + a_z^{321}d_3 + a_{xy}^{321}d_1d_2 + a_{yz}^{321}d_2d_3 + a_{xz}^{321}d_1d_3 + a_{xyz}^{321}d_1d_2d_3$

First we consider conditions under which expression (3.1) is meaningful. That the expression $\gamma_{123} \dot{\overline{\scriptstyle 1}}\, \gamma_{132}$ is well defined means the following five identities:

(3.11) $a_x^{123} = a_x^{132}$
(3.12) $a_y^{123} = a_y^{132}$
(3.13) $a_z^{123} = a_z^{132}$
(3.14) $a_{xy}^{123} = a_{xy}^{132}$
(3.15) $a_{xz}^{123} = a_{xz}^{132}$

And we have

(3.16) $(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132})(d_1, d_2, d_3) = a_x^{123}d_1 + (a_{yz}^{123} - a_{yz}^{132})d_2d_3 + (a_{xyz}^{123} - a_{xyz}^{132})d_1d_2d_3$

Similarly, that the expression $\gamma_{231} \underset{1}{\dot{-}} \gamma_{321}$ is well defined means the following five identities:

(3.17) $a_x^{231} = a_x^{321}$
(3.18) $a_y^{231} = a_y^{321}$
(3.19) $a_z^{231} = a_z^{321}$
(3.20) $a_{xy}^{231} = a_{xy}^{321}$
(3.21) $a_{xz}^{231} = a_{xz}^{321}$

And we have

(3.22) $(\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})(d_1, d_2d_3) = a_x^{231}d_1 + (a_{yz}^{231} - a_{yz}^{321})d_2d_3 + (a_{xyz}^{231} - a_{xyz}^{321})d_1d_2d_3$

That the expression $(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})$ is well defined means the following two identities:

(3.23) $a_x^{123} = a_x^{231}$
(3.24) $a_{yz}^{123} - a_{yz}^{132} = a_{yz}^{231} - a_{yz}^{321}$

And we have at last

(3.25) $((\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321}))(d_1d_2d_3) = (a_{xyz}^{123} - a_{xyz}^{132} - a_{xyz}^{231} + a_{xyz}^{321})d_1d_2d_3$

Now we turn to the expression (3.2). That the expression $\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}$ is well defined means the following five identities:

(3.26) $a_x^{231} = a_x^{213}$
(3.27) $a_y^{231} = a_y^{213}$
(3.28) $a_z^{231} = a_z^{213}$

(3.29) $a_{xy}^{231} = a_{xy}^{213}$
(3.30) $a_{yz}^{231} = a_{yz}^{213}$

And we have

(3.31) $(\gamma_{231} \div_2 \gamma_{213})(d_2, d_1d_3) = a_y^{231}d_2 + (a_{xz}^{231} - a_{xz}^{213})d_1d_3 + (a_{xyz}^{231} - a_{xyz}^{213})d_1d_2d_3$

Similarly, that the expression $\gamma_{312} \div_2 \gamma_{132}$ is well defined means the following five identities:

(3.32) $a_x^{312} = a_x^{132}$
(3.33) $a_y^{312} = a_y^{132}$
(3.34) $a_z^{312} = a_z^{132}$
(3.35) $a_{xy}^{312} = a_{xy}^{132}$
(3.36) $a_{yz}^{312} = a_{yz}^{132}$

And we have

(3.37) $(\gamma_{312} \div_2 \gamma_{132})(d_2, d_1d_3) = a_y^{312}d_2 + (a_{xz}^{312} - a_{xz}^{132})d_1d_3 + (a_{xyz}^{231} - a_{xyz}^{321})d_1d_2d_3$

That the expression $(\gamma_{231} \div_2 \gamma_{213}) \dot{-} (\gamma_{312} \div_2 \gamma_{132})$ is well defined means the following two identities:

(3.38) $a_y^{231} = a_y^{312}$
(3.39) $a_{xz}^{231} - a_{xz}^{213} = a_{xz}^{312} - a_{xz}^{132}$

And we have at last

(3.40) $((\gamma_{231} \div_2 \gamma_{213}) \dot{-} (\gamma_{312} \div_2 \gamma_{132}))(d_1d_2d_3) = (a_{xyz}^{231} - a_{xyz}^{213} - a_{xyz}^{312} + a_{xyz}^{132})d_1d_2d_3$

Now we turn to the expression (3.3). That the expression $\gamma_{312} \div_3 \gamma_{321}$ is well defined means the following five identities:

(3.41) $a_x^{312} = a_x^{321}$
(3.42) $a_y^{312} = a_y^{321}$
(3.43) $a_z^{312} = a_z^{321}$
(3.44) $a_{xz}^{312} = a_{xz}^{321}$
(3.45) $a_{yz}^{312} = a_{yz}^{321}$

And we have

(3.46) $(\gamma_{312} \div_3 \gamma_{321})(d_3, d_1d_2) = a_z^{312}d_3 + (a_{xy}^{312} - a_{xy}^{321})d_1d_2 + (a_{xyz}^{312} - a_{xyz}^{321})d_1d_2d_3$

Similarly, that the expression $\gamma_{123} \div_3 \gamma_{213}$ is well defined means the following five identities:

(3.47) $a_x^{123} = a_x^{213}$
(3.48) $a_y^{123} = a_y^{213}$
(3.49) $a_z^{123} = a_z^{213}$
(3.50) $a_{xz}^{123} = a_{xy}^{213}$
(3.51) $a_{yz}^{123} = a_{yz}^{213}$

And we have

(3.52) $(\gamma_{123} \dot{\overline{3}}\ \gamma_{213})(d_3, d_1d_2) = a_z^{123}d_3 + (a_{xy}^{123} - a_{xy}^{213})d_1d_2 + (a_{xyz}^{123} - a_{xyz}^{213})d_1d_2d_3$

That the expression $(\gamma_{312} \dot{\overline{3}}\ \gamma_{321}) \dot{-} (\gamma_{123} \dot{\overline{3}}\ \gamma_{213})$ is well defined means the following two identities:

(3.53) $a_z^{312} = a_z^{123}$
(3.54) $a_{xy}^{312} - a_{xy}^{321} = a_{xy}^{123} - a_{xy}^{213}$

And we have at last

(3.55) $((\gamma_{312} \dot{\overline{3}}\ \gamma_{321}) \dot{-} (\gamma_{123} \dot{\overline{3}}\ \gamma_{213}))(d_1d_2d_3) = (a_{xyz}^{312} - a_{xyz}^{321} - a_{xyz}^{123} + a_{xyz}^{213})d_1d_2d_3$

It is evident that the right-hand sides of (3.25), (3.40), and (3.55) sum up to 0.

As we have seen, in order that all of the three expressions (3.1)–(3.3) may be meaningful with γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312}, and γ_{321} in the forms (3.5)–(3.10) respectively, they must abide by 36 conditions (3.11)–(3.15), (3.17)–(3.21), (3.23), (3.24), (3.26)–(3.30), (3.32)–(3.36), (3.38), (3.39), (3.41)–(3.45), and (3.47)–(3.51), (3.53), and (3.54). Now we have to remark that these 36 conditions are far from independent. First note that five conditions (3.11), (3.17), (3.26), (3.32), and (3.41) can be put into the following form:

(3.56) $a_x^{123} = a_x^{132} = a_x^{213} = a_x^{231} = a_x^{312} = a_x^{321}$

The new condition (3.56) supersedes not only the above five conditions, but also conditions (3.23) and (3.47).

By the same token, the derivable condition

(3.57) $a_y^{123} = a_y^{132} = a_y^{213} = a_y^{231} = a_y^{312} = a_y^{321}$

supersedes the seven conditions (3.12), (3.18), (3.27), (3.33), (3.38), (3.42), and (3.48), and the derivable condition

(3.58) $a_z^{123} = a_z^{132} = a_z^{213} = a_z^{231} = a_z^{312} = a_z^{321}$

supersedes the seven conditions (3.13), (3.19), (3.28), (3.34), (3.43), (3.49), and (3.53).

Four conditions (3.14), (3.20), (3.29), and (3.35) can be recapitulated as follows:

(3.59) $a_{xy}^{123} = a_{xy}^{132} = a_{xy}^{312}$ and $a_{xy}^{213} = a_{xy}^{231} = a_{xy}^{321}$

We note that condition (3.54) is a direct consequence of the above condition.

Conditions (3.15), (3.21), (3.44), and (3.50) can be combined into

(3.60) $a_{xz}^{123} = a_{xz}^{132} = a_{xz}^{213}$ and $a_{xz}^{231} = a_{xz}^{312} = a_{xz}^{321}$

It is now evident that condition (3.39) is redundant.

Conditions (3.30), (3.36), (3.45), and (3.51) can be combined into

(3.61) $a_{yz}^{123} = a_{yz}^{213} = a_{yz}^{231}$ and $a_{yz}^{132} = a_{yz}^{312} = a_{yz}^{321}$

Now condition (3.24) readily turns out to be redundant.

The previous 36 conditions in a mess that the microcubes γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312}, and γ_{321} on R are required to satisfy have been replaced by the decent set of six conditions (3.56)–(3.61), of which we are no longer able to prune a modicum of superfluity.

What we have to do in order to convert the above simple argument of high school mathematics into a formal proof of Theorem 3.1 is, as is usual in synthetic differential geometry, to write out an appropriate quasi-colimit diagram of small objects corresponding to the above manipulation of polynomials, which we now present in the following lemma:

Lemma 3.3. The diagram consisting of objects

(3.62) 1
(3.63) E_{123}, E_{132}, E_{213}, E_{231}, E_{312}, E_{321}, all of which are equal to D^3
(3.64) E_{11}, E_{12}, both of which are equal to $D^3 \underset{1}{\vee} D$
(3.65) E_{21}, E_{22}, both of which are equal to $D^3 \underset{2}{\vee} D$
(3.66) E_{31}, E_{32}, both of which are equal to $D^3 \underset{3}{\vee} D$
(3.67) G_{11}, G_{12}, both of which are equal to $D[3; 2, 3]$
(3.68) G_{21}, G_{22}, both of which are equal to $D[3; 1, 3]$
(3.69) G_{31}, G_{32}, both of which are equal to $D[3; 1, 2]$
(3.70) G_1, G_2, G_3, all of which are equal to $D(2)$
(3.71) E, which is equal to $D^3 \underset{1}{\vee} D \underset{2}{\vee} D \underset{3}{\vee} D \vee D(2)$

and of morphisms

(3.72) $1 \xrightarrow{0} G_{11}$, $1 \xrightarrow{0} G_{12}$, $1 \xrightarrow{0} G_{21}$, $1 \xrightarrow{0} G_{22}$, $1 \xrightarrow{0} G_{31}$, $1 \xrightarrow{0} G_{32}$
(3.73) $G_{11} \xrightarrow{\iota} E_{123}$, $G_{11} \xrightarrow{\iota} E_{132}$, $G_{12} \xrightarrow{\iota} E_{231}$, $G_{12} \xrightarrow{\iota} E_{321}$, $G_{21} \xrightarrow{\iota} E_{231}$, $G_{21} \xrightarrow{\iota} E_{213}$, $G_{22} \xrightarrow{\iota} E_{312}$, $G_{22} \xrightarrow{\iota} E_{132}$, $G_{31} \xrightarrow{\iota} E_{312}$, $G_{31} \xrightarrow{\iota} E_{321}$, $G_{32} \xrightarrow{\iota} E_{123}$, $G_{32} \xrightarrow{\iota} E_{213}$

(3.74) $E_{132} \xrightarrow{\varphi_1} E_{11}$, $E_{321} \xrightarrow{\varphi_1} E_{12}$, $E_{213} \xrightarrow{\varphi_2} E_{21}$, $E_{132} \xrightarrow{\varphi_2} E_{22}$, $E_{321} \xrightarrow{\varphi_3} E_{31}$, $E_{213} \xrightarrow{\varphi_3} E_{32}$, $E_{123} \xrightarrow{\psi_1} E_{11}$, $E_{231} \xrightarrow{\psi_1} E_{12}$, $E_{231} \xrightarrow{\psi_2} E_{21}$, $E_{312} \xrightarrow{\psi_2} E_{22}$, $E_{312} \xrightarrow{\psi_3} E_{31}$, $E_{123} \xrightarrow{\psi_3} E_{32}$

(3.75) $G_1 \xrightarrow{\iota_1} E_{11}$, $G_1 \xrightarrow{\iota_1} E_{12}$, $G_2 \xrightarrow{\iota_2} E_{21}$, $G_2 \xrightarrow{\iota_2} E_{22}$, $G_3 \xrightarrow{\iota_3} E_{31}$, $G_3 \xrightarrow{\iota_3} E_{32}$

(3.76) $E_{11} \xrightarrow{\theta_{11}} E$, $E_{12} \xrightarrow{\theta_{12}} E$, $E_{21} \xrightarrow{\theta_{21}} E$, $E_{22} \xrightarrow{\theta_{22}} E$, $E_{31} \xrightarrow{\theta_{31}} E$, $E_{32} \xrightarrow{\theta_{32}} E$

is a quasi-colimit diagram of small objects with its quasi-colimit E, where

(3.77) $D^3 \underset{1}{\curlyvee} D \underset{2}{\curlyvee} D \underset{3}{\curlyvee} D \vee D(2) = \{(d_1, d_2, d_3, e_1, e_2, e_3, f_1, f_2) \in D^8 \mid e_1d_2 = e_1d_3 = e_2d_1 = e_2d_3 = e_3d_1 = e_3d_2 = e_1e_2 = e_1e_3 = e_2e_3 = f_1f_2 = f_1d_1 = f_1d_2 = f_1d_3 = f_1e_1 = f_1e_2 = f_1e_3 = f_2d_1 = f_2d_2 = f_2d_3 = f_2e_1 = f_2e_2 = f_2e_3 = 0\}$

(3.78) $\iota_1(d_1, d_2) = (d_1, 0, 0, d_2)$ for any $(d_1, d_2) \in D(2)$

(3.79) $\iota_2(d_1, d_2) = (0, d_1, 0, d_2)$ for any $(d_1, d_2) \in D(2)$

(3.80) $\iota_3(d_1, d_2) = (0, 0, d_1, d_2)$ for any $(d_1, d_2) \in D(2)$

(3.81) $\theta_{11}(d_1, d_2, d_3, d_4)$
$= (d_1, d_2, d_3, d_2d_3 - d_4, 0, 0, 0, 0)$
for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{1}{\curlyvee} D$

(3.82) $\theta_{12}(d_1, d_2, d_3, d_4)$
$= (d_1, d_2, d_3, d_2d_3 - d_4, d_1d_3, d_1d_2, 0, d_1d_2d_3 - d_1d_4)$
for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{1}{\curlyvee} D$

(3.83) $\theta_{21}(d_1, d_2, d_3, d_4)$
$= (d_1, d_2, d_3, 0, d_4, d_1d_2, 0, 0)$
for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{2}{\curlyvee} D$

(3.84) $\theta_{22}(d_1, d_2, d_3, d_4)$
$= (d_1, d_2, d_3, d_2d_3, d_4, 0, d_2d_4, 0)$
for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{2}{\curlyvee} D$

(3.85) $\theta_{31}(d_1, d_2, d_3, d_4)$
$= (d_1, d_2, d_3, d_2d_3, d_1d_3, d_1d_2 - d_4, d_3d_4, d_1d_2d_3 - d_3d_4)$
for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{3}{\curlyvee} D$

(3.86) $\theta_{32}(d_1, d_2, d_3, d_4)$
$= (d_1, d_2, d_3, 0, 0, d_1d_2 - d_4, 0, 0)$
for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{3}{\curlyvee} D$

Proof. The preceding considerations on the case of $M = \mathbb{R}$ show that the inverse limit of the diagram of objects (3.62)–(3.70) and morphisms (3.72)–(3.75) perceived by $\mathbb{R}$ can be identified with the set of 6-tuples (γ_{123},

$\gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321})$ of polynomials of d_1, d_2, and d_3 in D with coefficients in $\mathbb{R}$ of the following forms (3.87)–(3.92) that abide by conditions (3.56)–(3.61) and the succeeding condition (3.93):

(3.87) $\gamma_{123}(d_1, d_2, d_3) = a^{123} + a_x^{123}d_1 + a_y^{123}d_2 + a_z^{123}d_3 + a_{xy}^{123}d_1d_2 + a_{yz}^{123}d_2d_3 + a_{xz}^{123}d_1d_3 + a_{xyz}^{123}d_1d_2d_3$

(3.88) $\gamma_{132}(d_1, d_2, d_3) = a^{132} + a_x^{132}d_1 + a_y^{132}d_2 + a_z^{132}d_3 + a_{xy}^{132}d_1d_2 + a_{yz}^{132}d_2d_3 + a_{xz}^{132}d_1d_3 + a_{xyz}^{132}d_1d_2d_3$

(3.89) $\gamma_{213}(d_1, d_2, d_3) = a^{213} + a_x^{213}d_1 + a_y^{213}d_2 + a_x^{213}d_3 + a_{xy}^{213}d_1d_2 + a_{yz}^{213}d_2d_3 + a_{xz}^{213}d_1d_3 + a_{xyz}^{213}d_1d_2d_3$

(3.90) $\gamma_{231}(d_1, d_2, d_3) = a^{231} + a_x^{231}d_1 + a_y^{231}d_2 + a_z^{231}d_3 + a_{xy}^{231}d_1d_2 + a_{yz}^{231}d_2d_3 + a_{xz}^{231}d_1d_3 + a_{xyz}^{231}d_1d_2d_3$

(3.91) $\gamma_{312}(d_1, d_2, d_3) = a^{312} + a_x^{312}d_1 + a_y^{312}d_2 + a_z^{312}d_3 + a_{xy}^{312}d_1d_2 + a_{yz}^{312}d_2d_3 + a_{xz}^{312}d_1d_3 + a_{xyz}^{312}d_1d_2d_3$

(3.92) $\gamma_{321}(d_1, d_2, d_3) = a^{321} + a_x^{321}d_1 + a_y^{321}d_2 + a_z^{321}d_3 + a_{xy}^{321}d_1d_2 + a_{yz}^{321}d_2d_3 + a_{xz}^{321}d_1d_3 + a_{xyz}^{321}d_1d_2d_3$

(3.93) $a^{123} = a^{132} = a^{213} = a^{231} = a^{312} = a^{321}$

It is easy to see that for any function $\gamma: E \to \mathbb{R}$, the 6-tuple

$$(\gamma \circ \theta_{11} \circ \psi_1, \gamma \circ \theta_{11} \circ \varphi_1, \gamma \circ \theta_{21} \circ \varphi_2, \gamma \circ \theta_{12} \circ \psi_1, \gamma \circ \theta_{22} \circ \psi_2, \gamma \circ \theta_{12} \circ \varphi_1)$$

of functions from D^3 to $\mathbb{R}$ is identical with the 6-tuple

$$(\gamma \circ \theta_{32} \circ \psi_3, \gamma \circ \theta_{22} \circ \varphi_2, \gamma \circ \theta_{32} \circ \varphi_3, \gamma \circ \theta_{21} \circ \psi_2, \gamma \circ \theta_{31} \circ \psi_3, \gamma \circ \theta_{31} \circ \varphi_3)$$

of functions from D^3 to $\mathbb{R}$, and satisfies conditions (3.56)–(3.61) and (3.93) providing that the six functions are expressed as polynomials (3.87)–(3.92) in order. Conversely, given six polynomials (3.87)–(3.92) abiding by conditions (3.56)–(3.61) and (3.93), there exists exactly one function $\gamma: E \to \mathbb{R}$ such that the six functions of the 6-tuple

$$(\gamma \circ \theta_{11} \circ \psi_1, \gamma \circ \theta_{11} \circ \varphi_1, \gamma \circ \theta_{21} \circ \varphi_2, \gamma \circ \theta_{12} \circ \psi_1, \gamma \circ \theta_{22} \circ \psi_2, \gamma \circ \theta_{12} \circ \varphi_1)$$

are identical with polynomials (3.87)–(3.92) in order. More specifically, the desired γ should be a polynomial of d_1, d_2, d_3, e_1, e_2, e_3, f_1, and f_2 in D with coefficients in $\mathbb{R}$ of the following form:

(3.94) $\gamma(d_1, d_2, d_3, e_1, e_2, e_3, f_1 f_2) = a^{123} + a_x^{123}d_1 + a_y^{123}d_2 + a_z^{123}d_3 + a_{xy}^{123}d_1d_2 + a_{yz}^{123}d_2d_3 + a_{xz}^{123}d_1d_3 + a_{xyz}^{123}d_1d_2d_3 + (a_{yz}^{132} - a_{yz}^{123})e_1 + (a_{xz}^{231} - a_{xz}^{123})e_2 + (a_{xy}^{213} - a_{xy}^{123})e_3 + (a_{xyz}^{132} - a_{xyz}^{123})d_1e_1 + (a_{xyz}^{231} - a_{xyz}^{213})d_2e_2 + (a_{xyz}^{213} - a_{xyz}^{123})d_3e_3 + (a_{xyz}^{312} - a_{xyz}^{132} + a_{xyz}^{213} - a_{xyz}^{231})f_1 + (a_{xyz}^{321} - a_{xyz}^{132} + a_{xyz}^{123} - a_{xyz}^{231})f_2$

These considerations show that the assignment, to each function $\gamma: E \to \mathbb{R}$, of the 6-tuple

$$(\gamma \circ \theta_{11} \circ \psi_1, \gamma \circ \theta_{11} \circ \varphi_1, \gamma \circ \theta_{21} \circ \varphi_2, \gamma \circ \theta_{12} \circ \psi_1, \gamma \circ \theta_{22} \circ \psi_2, \gamma \circ \theta_{12} \circ \varphi_1)$$

renders a bijective correspondence between the functions from E to $\mathbb{R}$ and the 6-tuples of polynomials of forms (3.87)–(3.92) abiding by conditions (3.56)–(3.61) and (3.93). Therefore the proof is complete. ∎

Proposition 3.4. If all three expressions (3.1)–(3.3) are meaningful, there exists a unique function

$$\mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} : D^3 \underset{1}{\vee} D \underset{2}{\vee} D \underset{3}{\vee} D \vee D(2) \to M$$

abiding by the following conditions:

$$\begin{aligned}
&(3.95) \quad \mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} \circ \theta_{11} = \mathscr{g}^{1}_{(\gamma_{132},\gamma_{123})} \\
&(3.96) \quad \mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} \circ \theta_{12} = \mathscr{g}^{1}_{(\gamma_{321},\gamma_{231})} \\
&(3.97) \quad \mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} \circ \theta_{21} = \mathscr{g}^{2}_{(\gamma_{213},\gamma_{312})} \\
&(3.98) \quad \mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} \circ \theta_{22} = \mathscr{g}^{2}_{(\gamma_{132},\gamma_{312})} \\
&(3.99) \quad \mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} \circ \theta_{31} = \mathscr{g}^{3}_{(\gamma_{321},\gamma_{312})} \\
&(3.100) \quad \mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} \circ \theta_{32} = \mathscr{g}^{3}_{(\gamma_{213},\gamma_{123})}
\end{aligned}$$

Proof. Follows from Lemma 3.3. ∎

Now we are ready to present a proof of Theorem 3.1.

Proof of Theorem 3.1. Assuming that all three expressions (3.1)–(3.3) are meaningful, let

$$\mathscr{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})} : D^3 \underset{1}{\vee} D \underset{2}{\vee} D \underset{3}{\vee} D \vee D(2) \to M$$

be the function in Proposition 3.4, which we now denote simply by $\mathscr{m}$. It follows from condition (3.95) that for any $(d_1, d_2) \in D^2$,

$$(3.101) \quad (\gamma_{123} \underset{1}{\dot{-}} \gamma_{132})(d_1, d_2) = \mathscr{g}^{1}_{(\gamma_{132},\gamma_{123})}(d_1, 0, 0, d_2) = \mathscr{m}(d_1, 0, 0, -d_2, 0, 0, 0, 0)$$

It follows from condition (3.96) that for any $(d_1, d_2) \in D^2$,

$$(3.102) \quad (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321})(d_1, d_2) = \mathscr{g}^{1}_{(\gamma_{321},\gamma_{231})}(d_1, 0, 0, d_2) = \mathscr{m}(d_1, 0, 0, -d_2, 0, 0, 0, -d_1 d_2)$$

It follows from condition (3.97) that for any $(d_1, d_2) \in D^2$,

$$(3.103) \quad (\gamma_{231} \underset{2}{\dot{-}} \gamma_{213})(d_1, d_2) = \mathscr{g}^{2}_{(\gamma_{213},\gamma_{231})}(0, d_1, 0, d_2) = \mathscr{m}(0, d_1, 0, 0, d_2, 0, 0, 0)$$

It follows from condition (3.98) that for any $(d_1, d_2) \in D^2$,

$$(3.104) \quad (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132})(d_1, d_2) = \mathscr{g}^{2}_{(\gamma_{132},\gamma_{312})}(0, d_1, 0, d_2) = \mathscr{m}(0, d_1, 0, 0, d_2, 0, d_1 d_2, 0)$$

It follows from condition (3.99) that for any $(d_1, d_2) \in D^2$,

$$(3.105)\quad (\gamma_{312} \underset{3}{\dot{-}} \gamma_{321})(d_1, d_2) =_{\mathscr{g}^3_{(\gamma_{321},\gamma_{312})}}(0, 0, d_1, d_2)$$
$$=_{\mathscr{m}}(0, 0, d_1, 0, 0, -d_2, d_1d_2, -d_1d_2)$$

It follows from condition (3.100) that for any $(d_1, d_2) \in D^2$,

$$(3.106)\quad (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213})(d_1, d_2) =_{\mathscr{g}^3_{(\gamma_{213},\gamma_{123})}}(0, 0, d_1, d_2)$$
$$=_{\mathscr{m}}(0, 0, d_1, 0, 0, -d_2, 0, 0)$$

It follows from (3.101) and (3.102) that for any $(d_1, d_2, d_3) \in D^2 \vee D$,

$$(3.107)\quad \mathscr{g}_{(\gamma_{231}\underset{1}{\dot{-}}\gamma_{321},\gamma_{123}\underset{1}{\dot{-}}\gamma_{132})}(d_1, d_2, d_3)$$
$$=_{\mathscr{m}}(d_1, 0, 0, -d_2, 0, 0, 0, d_3 - d_1d_2)$$

It follows from (3.103) and (3.104) that for any $(d_1, d_2, d_3) \in D^2 \vee D$,

$$(3.108)\quad \mathscr{g}_{(\gamma_{312}\underset{2}{\dot{-}}\gamma_{132},\gamma_{231}\underset{2}{\dot{-}}\gamma_{213})}(d_1, d_2, d_3)$$
$$=_{\mathscr{m}}(0, d_1, 0, 0, d_2, 0, d_1d_2 - d_3, 0)$$

It follows from (3.105) and (3.106) that $(d_1, d_2, d_3) \in D^2 \vee D$,

$$(3.109)\quad \mathscr{g}_{(\gamma_{123}\underset{3}{\dot{-}}\gamma_{213},\gamma_{312}\underset{3}{\dot{-}}\gamma_{321})}(d_1, d_2, d_3)$$
$$=_{\mathscr{m}}(0, 0, d_1, 0, 0, -d_2, d_3, -d_3)$$

It follows from (3.107) that for any $d \in D$,

$$(3.110)\quad ((\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321}))(d)$$
$$=_{\mathscr{m}}(0, 0, 0, 0, 0, 0, 0, d)$$

It follows from (3.108) that for any $d \in D$,

$$(3.111)\quad ((\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132}))(d)$$
$$=_{\mathscr{m}}(0, 0, 0, 0, 0, 0, -d, 0)$$

It follows from (3.109) that for any $d \in D$,

$$(3.112)\quad ((\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213}))(d)$$
$$=_{\mathscr{m}}(0, 0, 0, 0, 0, 0, d, -d)$$

It follows readily from (3.110)–(3.112) that for any $(d_1, d_2, d_3) \in D(3)$,

$$(3.113)\quad l_{(t_1,t_2,t_3)}(d_1, d_2, d_3)$$
$$=_{\mathscr{m}}(0, 0, 0, 0, 0, 0, -d_2 + d_3, d_1 - d_3)$$

where t_1, t_2, and t_3 stand for tangent vectors represented by (3.1), (3.2), and (3.3), respectively. Therefore, for any $d \in D$,

$$(3.114)\quad (t_1 + t_2 + t_3)(d)$$
$$= l_{(t_1,t_2,t_3)}(d, d, d)$$
$$=_{\mathscr{m}}(0, 0, 0, 0, 0, 0, 0, 0)$$

Now the proof is complete. ∎

4. SIMPLICIAL OBJECTS

Let n be a natural number and $\mathbf{n}$ the set consisting exactly of $1, 2, \ldots, n$. For any natural number k with $2 \leq k \leq n$, we let

$$(4.1) \quad \Delta_n^k = \{(i_1, \ldots, i_k) \in \mathbf{n}^k \mid i_1 < \cdots < i_k\}$$

We let $\Delta_n = \cup_{k=2}^n \Delta_n^k$. For any subset $\mathfrak{p} \subset \Delta_n$, we define

$$(4.2) \quad D^n(\mathfrak{p}) = \{(d_1, d_2, \ldots, d_n) \in D^n \mid d_{i_1} \cdots d_{i_k} = 0 \text{ for any } (i_1, \ldots, i_k) \in \mathfrak{p}\}$$

If $\mathfrak{p}$ is the empty set, $D^n(\mathfrak{p})$ is D^n itself. If $\mathfrak{p} = \Delta_n$, then $D^n(\mathfrak{p})$ is $D(n)$ in standard terminology. A small object of the form $D^n(\mathfrak{p})$ for some subset $\mathfrak{p}$ of Δ_n is called a *simplicial object of degree n*. The small object $D(n)$ is called the *tensorial object of degree n*. Given $(i, j) \in \Delta_n$, we denote the set $D^n(\{(i, j)\})$ by $D[n; i, j]$, which is compatible with the notation of Section 2. Given a simplicial object $D^n(\mathfrak{p})$, we denote by $\mathsf{S}_{D^n(\mathfrak{p})}(M; m)$ the set of functions τ: $D^n(\mathfrak{p}) \to M$ with $\tau(0, \ldots, 0) = m$. We denote by $\mathsf{S}_{D^n(\mathfrak{p})}(M)$ the set of all functions τ: $D^n(\mathfrak{p}) \to M$. We denote by $\imath_i$ the ith injection of D into $D^n(\mathfrak{p})$ $(1 \leq i \leq n)$. The function

$$\tau \in \mathsf{S}_{D^n(\mathfrak{p})}(M; m) \mapsto (\tau \circ \imath_1, \ldots, \tau \circ \imath_n) \in (\mathsf{T}^1(M; m))^n$$

which is the restriction function to the axes, is generally denoted by K.

Now we make explicit a result on $D(2)$, which should be considered to belong to the folklore of synthetic differential geometry.

Lemma 4.1. The diagram

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "0"'] & D \arrow[d, "\imath_2"] \\ D \arrow[r, "\imath_1"'] & D(2) \end{tikzcd}$$

is a quasi-colimit diagram of small objects, where

(4.3) $\imath_1(d) = (d, 0)$ for any $d \in D$;
(4.4) $\imath_2(d) = (0, d)$ for any $d \in D$.

Proof. See Proposition 6 of Lavendhomme (1996, §2.2). ∎

Proposition 4.2. For any $t_1, t_2 \in \mathsf{T}^1(M; m)$, there exists a unique function α: $D(2) \to M$ such that $\alpha(d, 0) = t_1(d)$ and $\alpha(0, d) = t_2(d)$ for any $d \in D$, in which we write $\alpha = t_1 + t_2$.

Proof. Follows from Lemma 4.1. ■

Therefore K: $\mathsf{S}_{D(2)}(M; m) \to \mathsf{T}^1(M; m) \times \mathsf{T}^1(M; m)$ is a bijection. By the same token, K: $\mathsf{S}_{D(n)}(M; m) \to (\mathsf{T}^1(M; m))^n$ is a bijection for any natural number n.

The following lemma and proposition are generalizations of the preceding lemma and proposition on $D(2)$ to $D[n + 2; n + 1, n + 2]$.

Lemma 4.3. The diagram

$$\begin{tikzcd} D^n \arrow[r, "i"] \arrow[d, "i"'] & D^{n+1} \arrow[d, "k_2"] \\ D^{n+1} \arrow[r, "k_1"'] & D[n+2; n+1, n+2] \end{tikzcd}$$

is a quasi-colimit diagram of small objects, where

(4.5) $i(d_1, \ldots, d_n) = (d_1, \ldots, d_n, 0)$ for any $(d_1, \ldots, d_n) \in D^n$ (unless stated to the contrary, D^n is regarded canonically as a subobject of D^{n+1} under the injection i);

(4.6) $k_1(d_1, \ldots, d_{n+1}) = (d_1, \ldots, d_{n+1}, 0)$ for any $(d_1, \ldots, d_{n+1}) \in D^{n+1}$;

(4.7) $k_2(d_1, \ldots, d_{n+1}) = (d_1, \ldots, d_n, 0, d_{n+1})$ for any $(d_1, \ldots, d_{n+1}) \in D^{n+1}$.

Proposition 4.4. For any $\delta_1, \delta_2 \in \mathsf{T}^{n+1}(M; m)$ with $\delta_1 \circ i = \delta_2 \circ i$, there exists a unique function τ: $D[n + 2; n + 1, n + 2] \to M$ such that $\tau \circ k_1 = \delta_1$ and $\tau \circ k_2 = \delta_2$, in which we write $\tau = \delta_1 \oplus \delta_2$.

Proof. Follows from Lemma 4.3. ■

Given $\delta_1, \delta_2 \in \mathsf{T}^{n+1}(M; m)$ with $\delta_1|_{D^n} = \delta_2|_{D^n}$, we define a function $\nabla(\delta_1, \delta_2)$: $D^{n+2} \to M$ such that for any $(d_1, \ldots, d_{n+2}) \in D^{n+2}$.

(4.8) $\nabla(\delta_1, \delta_2)(d_1, \ldots, d_{n+2}) = \nabla(\delta_1(d_1, \ldots, d_n), \delta_2(d_1, \ldots, d_n))(d_{n+1}, d_{n+2})$

where in the right-hand side of the equality δ_1 and δ_2 are to be regarded as functions from D^n to M^D in the expected manner.

Given $(i, j) \in \Delta^2_{n+2}$ and $\tau \in \mathsf{T}^{n+2}(M; m)$, we define $\tilde{\nabla}_{i,j}\tau \in \mathsf{T}^{n+2}(M; m)$ and $\tilde{\nabla}_{j,i}\tau \in \mathsf{T}^{n+2}(M; m)$, also written $\tilde{\nabla}_{ij}\tau$ and $\tilde{\nabla}_{ji}\tau$, as follows:

(4.9) In case of $(i, j) = (n + 1, n + 2)$, let $\delta_1, \delta_2 \in \mathsf{T}^{n+1}(M; m)$ such that $\delta_1 + \delta_2 = \tau|_{D[n+2;n+1,n+2]}$; we define $\tilde{\nabla}_{n+1,n+2}\tau = \nabla(\delta_2, \delta_2)$.

(4.10) In case of $j = n + 1$, we define $\tilde{\nabla}_{ij}\tau = \Sigma_{(i,n+2)}(\Sigma_{(n+1,n+2)}(\tilde{\nabla}_{n+1,n+2}\Sigma_{(i,n+1)}(\Sigma_{(n+1,n+2)}(\tau)))$

(4.11) In case of $j \le n$, we define $\tilde{\nabla}_{ij}\tau = \Sigma_{(i,n+1)}(\Sigma_{(j,n+2)}(\tilde{\nabla}_{n+1,n+2}\Sigma_{(i,n+1)}(\Sigma_{(j,n+2)}(\tau)))))$

(4.12) We define $\tilde{\nabla}_{ji}\tau = \Sigma_{(i,j)}(\tilde{\nabla}_{ij}\Sigma_{(i,j)}(\tau))$.

In case of $n = 0$ and $(i, j) = (1, 2)$, $\tilde{\nabla}_{ij}\tau$ is often written simply $\tilde{\nabla}\tau$.

Proposition 4.5. Let n and k be natural numbers. Let $t_i \in \mathsf{T}^1(M; m)$ $(1 \le i \le n + k)$. Then there exist $\tau_j \in \mathsf{T}^{n+1}(M; m)$ $(1 \le j \le k)$ such that

(4.13) $\tau_1|_{D^n} = \cdots = \tau_k|_{D^n}$;
(4.14) $\tau_j \circ \iota_i = t_i$ $(1 \le j \le k,\ 1 \le i \le n)$;
(4.15) $\tau_j \circ \iota_{n+1} = t_{n+j}$ $(1 \le j \le k)$.

Proof. The proof is carried out by induction on n. If $n = 0$, the theorem holds trivially irrespective of k. Now we show that, assuming that the theorem holds for a pair (n, k) of natural numbers with arbitrary k, the theorem holds for $(n + 1, k)$. Let $t_i \in \mathsf{T}^1(M; m)$ $(1 \le i \le n + k + 1)$. By assumption the theorem holds for $(n, k + 1)$, so that there exist $\delta, \bar{\tau}_j \in \mathsf{T}^{n+1}(M; m)$ $(1 \le j \le k)$ such that

(4.16) $\delta|_{D^n} = \bar{\tau}_1|_{D^n} = \cdots = \bar{\tau}_k|_{D^n}$;
(4.17) $\delta \circ \iota_1 = \bar{\tau}_j \circ \iota_i = t_i$ $(1 \le j \le k,\ 1 \le i \le n)$;
(4.18) $\delta \circ \iota_{n+1} = t_{n+1}$;
(4.19) $\bar{\tau}_j \circ \iota_{n+1} = t_{n+j+1}$ $(1 \le j \le k)$.

We now take $\nabla(\delta, \bar{\tau}_1), \ldots, \nabla(\delta, \bar{\tau}_k)$ for $\tau_1, \ldots, \tau_k$, which are easily seen to satisfy conditions (4.13)–(4.15). ■

Corollary 4.6. For any function $\delta: D(n) \to M$, there exists a function $\tau: D^n \to M$ such that $\tau|_{D(n)} = \delta$.

Proof. Since $K: \mathsf{S}_{D(n)}(M; m) \to (\mathsf{T}^1(M; m))^n$ is a bijection, the desired result follows from the above theorem in case of $k = 1$. ■

Given a simplicial object $D^n(\mathfrak{p})$ of degree n, a (*simplicial*) $D^n(\mathfrak{p})$-*form* is a function $\omega: \mathsf{S}_{D^n(\mathfrak{p})}(M) \to R$ such that ω is homogeneous componentwise. Alternating D^n-forms are called *singular differential forms of degree n*, while alternating $D(n)$-forms are called *classical differential forms of degree n*. Given two simplicial objects $D^n(\mathfrak{p})$ and $D^n(\mathfrak{q})$ of the same degree n with $D^n(\mathfrak{q}) \subset D^n(\mathfrak{p})$, a $D^n(\mathfrak{p})$-form ω is said to be *essentially* a $D^n(\mathfrak{q})$-form if for any $\tau_1, \tau_2 \in \mathsf{S}_{D^n(\mathfrak{p})}(M; m)$ with $\tau_1|_{D^n(\mathfrak{q})} = \tau_2|_{D^n(\mathfrak{q})}$, we have $\omega(\tau_1) = \omega(\tau_2)$. In particular, a $D^n(\mathfrak{p})$-form is said to be *essentially tensorial* if it is essentially a $D(n)$-form.

Proposition 4.7. Singular differential forms of degree n are essentially classical differential forms of degree n.

Proof. By the same token as Lavendhomme (1966, §4.1, Proposition 5). ■

Corollary 4.8. Singular differential forms of degree n and classical differential forms of degree n are in bijective correspondence.

Proof. This follows from Corollary 4.6 and Proposition 4.7. ■

In passing we note that, to establish Corollary 4.8, we do not need such an eccentric notion as that of symmetrical 3-connection discussed by Lavendhomme (1996, §4.1.2). Our Corollary 4.6 will do. We also note that the notion of $D^n(\mathfrak{p})$-form can be generalized easily to that of $D^n(\mathfrak{p})$-form with values in a tangent fiber bundle, as Lavendhomme (1996, §5.3.1) did in differential forms, and Proposition 4.7 and Corollary 4.8 hold with due modifications. In particular, the curvature form Ω is a D^3-form in this extended sense (more specifically it is a $D[3; 1, 2]$-form in the extended sense), and it will be shown in Section 3 that it is essentially tensorial.

5. ANOTHER STRONG DIFFERENCE IN MICROCUBES

The following lemma is another three-dimensional generalization of Lemma 1.1.

Lemma 5.1. The diagram

$$\begin{tikzcd} D[3] \arrow[r, "\iota"] \arrow[d, "\iota"'] & D^3 \arrow[d, "\overline{\psi}"] \\ D^3 \arrow[r, "\overline{\varphi}"'] & D^3 \vee D \end{tikzcd}$$

is a quasi-colimit diagram of small objects, where

(5.1) $D[3] = D^3(\{(1, 2, 3)\})$;
(5.2) $D^3 \vee D = D^4(\{(1, 4), (2, 4), (3, 4)\})$;
(5.3) $\overline{\varphi}(d_1, d_2, d_3) = (d_1, d_2, d_3, 0)$ for any $(d_1, d_2, d_3) \in D^3$;
(5.4) $\overline{\psi}(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1d_2d_3)$ for any $(d_1, d_2, d_3) \in D^3$.

The lemma enables us to define the notion of strong difference for microsquares as follows:

Proposition 5.2. For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M; m)$, if $\gamma_1|_{D[3]} = \gamma_2|_{D[3]}$, then there exists a unique function $\bar{\mathscr{g}}_{(\gamma_2,\gamma_2)} \cdot \varphi = \gamma_1$ and $\bar{\mathscr{g}}_{(\gamma_1,\gamma_2)} \cdot \psi = \gamma_2$. In this case we define a tangent vector $\gamma_2 \dot{-} \gamma_1$ in $\mathsf{T}^1(M; m)$ as follows:

(5.5) $(\gamma_2 \dot{-} \gamma_1)(d) = \bar{\mathscr{g}}_{(\gamma_1,\gamma_2)}(0, 0, 0, d)$ for any $d \in D$.

We could define the notion of strong translation and proceed similarly as in Lavendhomme (1996, §3.4), the details of which are safely left to the reader. We note only the following:

Proposition 5.3. Let $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$ and $\rho \in \mathfrak{Perm}_3$. Then $\gamma_1|_{D[3]} = \gamma_2|_{D[3]}$ iff $\Sigma_\rho(\gamma_1)|_{D[3]} = \Sigma_\rho(\gamma_2)|_{D[3]}$, in which

$$(5.6)\quad \gamma_2 \dot{-} \gamma_1 = \Sigma_\rho(\gamma_2) \dot{-} \Sigma_\rho(\gamma_1)$$

The following theorem is in the same vein as the general Jacobi identity theorem, but the proof of the former is much easier than that of the latter.

Theorem 5.4. For any $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{T}^3(M; m)$, if all of the following three expressions

$$(5.7)\quad \gamma_1 \dot{-} \gamma_2$$
$$(5.8)\quad \gamma_2 \dot{-} \gamma_3$$
$$(5.9)\quad \gamma_3 \dot{-} \gamma_1$$

are well defined, they sum up only to vanish.

To prove the above theorem, we need, first of all, the following lemma:

Lemma 5.5. The diagram consisting of objects

(5.10) 1
(5.11) G_{11}, G_{13}, G_{23}, all of which are equal to $D[3]$
(5.12) E_1, E_2, E_3, all of which are equal to D^3
(5.13) E, which is equal to $D^2 \vee D(2)$

and consisting of morphisms

$$(5.14)\quad 1 \xrightarrow{0} G_{12},\ 1 \xrightarrow{0} G_{13},\ 1 \xrightarrow{0} G_{23}$$
$$(5.15)\quad G_{12} \xrightarrow{\iota} E_1,\ G_{12} \xrightarrow{\iota} E_2,\ G_{23} \xrightarrow{\iota} E_2,\ G_{23} \xrightarrow{\iota} E_3,\ G_{31} \xrightarrow{\iota} E_3,\ G_{31} \xrightarrow{\iota} E_1$$
$$(5.16)\quad E_1 \xrightarrow{\theta_1} E,\ E_2 \xrightarrow{\theta_2} E,\ E_3 \xrightarrow{\theta_3} E$$

Is a quasi-colimit diagram of small objects with its quasi-colimit E, where

$$(5.17)\quad D^3 \vee D(2) = D^5(\{(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),(4,5)\}$$
$$(5.18)\quad \theta_1(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0) \text{ for any } (d_1, d_2, d_3) \in D^3$$
$$(5.19)\quad \theta_2(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1d_2d_3, 0) \text{ for any } (d_1, d_2, d_3) \in D^3$$
$$(5.20)\quad \theta_3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, d_1d_2d_3) \text{ for any } (d_1, d_2, d_3) \in D^3$$

Proof. It is easy to see that the inverse limit of the diagram of objects (5.10)–(5.12) and morphisms (5.14) and (5.15) perceived by R can naturally be identified with the set of triples $(\gamma_1, \gamma_2, \gamma_3)$ of polynomials of d_1, d_2, and d_3 in D with coefficients in R of the following forms (5.21)–(5.23) abiding by conditions (5.24)–(5.30):

$$(5.21)\quad \gamma_1(d_1, d_2, d_3) = a^1 + a_x^1 d_1 + a_y^1 d_2 + a_z^1 d_3 + a_{xy}^1 d_1 d_2 + a_{yz}^1 d_2 d_3 + a_{xz}^1 d_1 d_3 + a_{xyz}^1 d_1 d_2 d_3$$
$$(5.22)\quad \gamma_2(d_1, d_2, d_3) = a^2 + a_x^2 d_1 + a_y^2 d_2 + a_z^2 d_3 + a_{xy}^2 d_1 d_2 + a_{yz}^2 d_2 d_3 + a_{xz}^2 d_1 d_3 + a_{xyz}^2 d_1 d_2 d_3$$
$$(5.23)\quad \gamma_3(d_1, d_2, d_3) = a^3 + a_x^3 d_1 + a_y^3 d_2 + a_z^3 d_3 + a_{xy}^3 d_1 d_2 + a_{yz}^3 d_2 d_3 + a_{xz}^3 d_1 d_3 + a_{xyz}^3 d_1 d_2 d_3$$
$$(5.24)\quad a^1 = a^2 = a^3$$
$$(5.25)\quad a_x^1 = a_x^2 = a_x^3$$
$$(5.26)\quad a_y^1 = a_y^2 = a_y^3$$
$$(5.27)\quad a_z^1 = a_z^2 = a_z^3$$
$$(5.28)\quad a_{xy}^1 = a_{xy}^2 = a_{xy}^3$$
$$(5.29)\quad a_{yz}^1 = a_{yz}^2 = a_{yz}^3$$
$$(5.30)\quad a_{xz}^1 = a_{xz}^2 = a_{xz}^3$$

It is easy to see that for any function γ: $E \to \mathbf{R}$, the triple $(\gamma \circ \theta_1, \gamma \circ \theta_2, \gamma \circ \theta_3)$ of functions from D^3 to $\mathbf{R}$ satisfies conditions (5.24)–(5.30) provided that the three functions are expressed as polynomials (5.21)–(5.23) in order. Conversely, given three polynomials (5.21)–(5.23) satisfying conditions (5.24)–(5.30), there exists exactly one function γ: $E \to \mathbf{R}$ such that the three functions in the triple $(\gamma \circ \theta_1, \gamma \circ \theta_2, \gamma \circ \theta_3)$ are identical with polynomials (5.21)–(5.23) in order. More specifically, the desired γ should be a polynomial of d_1, d_2, d_3, e_1, e_2 in D with coefficients in $\mathbf{R}$ of the following form:

$$(5.31)\quad \gamma(d_1, d_2, d_3, e_1, e_2) = a^1 + a_x^1 d_1 + a_y^1 d_2 + a_z^1 d_3 + a_{xy}^1 d_1 d_2 + a_{yz}^1 d_2 d_3 + a_{xz}^1 d_1 d_3 + a_{xyz}^1 d_1 d_2 d_3 + (a_{xyz}^2 - a_{xyz}^1) e_1 + (a_{xyz}^3 - a_{xyz}^1) e_2$$

These considerations show that the assignment, to each function γ: $E \to \mathbf{R}$, of the triple $(\gamma \circ \theta_1, \gamma \circ \theta_2, \gamma \circ \theta_3)$ renders a bijective correspondence between the functions from E to $\mathbf{R}$ and the triples of polynomials of forms (5.21)–(5.23) satisfying conditions (5.24)–(5.30). Therefore the proof is complete. ■

Proposition 5.6. If all of the three expressions (5.7)–(5.9) are meaningful, there exists a unique function $\mathscr{n}_{(\gamma_1,\gamma_2,\gamma_3)}$: $D^3 \vee D(2) \to M$ such that

$$(5.32)\quad \mathscr{n}_{(\gamma_1,\gamma_2,\gamma_3)} \circ \theta_i = \gamma_i \ (i = 1, 2, 3).$$

Proof. Follows from Lemma 5.5. ■

Now we are ready to present a proof of Theorem 5.4.

Proof of Theorem 5.4. We denote $\mathscr{n}_{(\gamma_1,\gamma_2,\gamma_3)}$ by $\mathscr{n}$ for simplicity. The condition (5.21) means that for any $(d_1, d_2, d_3) \in D^3$, we have the following:

$$(5.33)\quad \gamma_1(d_1, d_2, d_3) = \mathscr{n}(d_1, d_2, d_3, 0, 0)$$
$$(5.34)\quad \gamma_2(d_1, d_2, d_3) = \mathscr{n}(d_1, d_2, d_3, d_1 d_2 d_3, 0)$$

$$(5.35)\quad \gamma_3(d_1, d_2, d_3) = {\mathscr n}(d_1, d_2, d_3, 0, d_1d_2d_3)$$

It follows from (5.33) and (5.34) that

$$(5.36)\quad \bar{\mathscr g}_{(\gamma_2,\gamma_1)}(d_1, d_2, d_3, d_4) = {\mathscr n}(d_1, d_2, d_3, d_1d_2d_3 - d_4, 0)$$

for any $(d_1, d_2, d_3, d_4) \in D^2 \vee D$. It follows from (5.34) and (5.35) that

$$(5.37)\quad \bar{\mathscr g}_{(\gamma_3,\gamma_2)}(d_1\ d_2, d_3, d_4) = {\mathscr n}(d_1, d_2, d_3, d_4, d_1d_2d_3 - d_4)$$

for any $(d_1, d_2, d_3, d_4) \in D^3 \vee D$. It follows from (5.33) and (5.35) that

$$(5.38)\quad \bar{\mathscr g}_{(\gamma_1,\gamma_3)}(d_1, d_2, d_3, d_4) = {\mathscr n}(d_1, d_2, d_3, 0, d_4)$$

for any $(d_1, d_2, d_3, d_4) \in D^3 \vee D$. It follows from (5.36) that

$$(5.39)\quad (\gamma_1 \dot{-} \gamma_2)(d) = {\mathscr n}(0, 0, 0, -d, 0)$$

for any $d \in D$. It follows from (5.37) that

$$(5.40)\quad (\gamma_2 \dot{-} \gamma_3)(d) = {\mathscr n}(0, 0, 0, d, -d)$$

for any $d \in D$. It follows from (5.38) that

$$(5.41)\quad (\gamma_3 \dot{-} \gamma_1)(d) = {\mathscr n}(0, 0, 0, 0, d)$$

for any $d \in D$. It follows from (5.39)–(5.41) that

$$(5.42)\quad l_{(t_1,t_2,t_3)}(d_1, d_2, d_3) = {\mathscr n}(0, 0, 0, d_2 - d_1, -d_2 + d_3)$$

for any $(d_1, d_2, d_3) \in D(3)$. Therefore, for any $d \in D$, we have

$$(5.43)\quad (t_1 + t_2 + t_3)(d) = l_{(t_1,t_2,t_3)}(d, d, d) = {\mathscr n}(0, 0, 0, 0, 0)$$

Now the proof is complete. ■

We conclude this section with some minor results which will be needed in the next section. The first result, relating the strong difference of this section to those of Section 2, is reminiscent of Proposition 7 of Lavendhomme (1996, §3.4).

Proposition 5.7. Let $\gamma_1, \gamma_2 \in \mathrm{T}^3(M; m)$ with $\gamma_1|_{D[3]} = \gamma_2|_{D[3]}$. Then

$$(5.44)\quad \begin{aligned} \gamma_2 \dot{-} \gamma_1 &= (\gamma_2 \underset{1}{\dot{-}} \gamma_1) \dot{-} \tilde{t}_1 \\ &= (\gamma_2 \underset{2}{\dot{-}} \gamma_1) \dot{-} \tilde{t}_2 \\ &= (\gamma_2 \underset{3}{\dot{-}} \gamma_1) \dot{-} \tilde{t}_3 \end{aligned}$$

where for any $(d_1, d_2) \in D^2$,

$$\begin{aligned} &(5.45)\quad \tilde{t}_1(d_1, d_2) = \gamma_1(d_1, 0, 0) = \gamma_2(d_1, 0, 0) \\ &(5.46)\quad \tilde{t}_2(d_1, d_2) = \gamma_1(0, d_1, 0) = \gamma_2(0, d_1, 0) \\ &(5.47)\quad \tilde{t}_3(d_1, d_2) = \gamma_1(0, 0, d_1) = \gamma_2(0, 0, d_1) \end{aligned}$$

Proof. Here we deal only with the first equality, leaving similar treatments of the other two equalities to the reader. Let $\bar{\mathscr{g}} = \bar{\mathscr{g}}_{(\gamma_1,\gamma_2)}$: $D^2 \vee D \to M$ be the function such that for any $(d_1, d_2, d_3) \in D^3$,

$$(5.48) \quad \gamma_1(d_1, d_2, d_3) = \bar{\mathscr{g}}(d_1, d_2, d_3, 0)$$
$$(5.49) \quad \gamma_2(d_1, d_2, d_3) = \bar{\mathscr{g}}(d_1, d_2, d_3, d_1d_2d_3)$$

Then for any $d \in D$,

$$(5.50) \quad (\gamma_2 \dot{-} \gamma_1)(d) = \bar{\mathscr{g}}(0, 0, 0, d)$$

Let $\mathscr{k}_1$: $D^3 \underset{1}{\vee} D \to M$ be the function such that for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{1}{\vee} D$,

$$(5.51) \quad \mathscr{k}_1(d_1, d_2, d_3, d_4) = \bar{\mathscr{g}}(d_1, d_2, d_3, d_1d_4)$$

Then it follows from (5.48) and (5.49) that for any $(d_1, d_2, d_3) \in D^3$,

$$(5.52) \quad \gamma_1(d_1, d_2, d_3) = \mathscr{k}_1(d_1, d_2, d_3, 0)$$
$$(5.53) \quad \gamma_2(d_1, d_2, d_3) = \mathscr{k}_1(d_1, d_2, d_3, d_2d_3)$$

Therefore for any $(d_1, d_4) \in D^2$,

$$(5.54) \quad (\gamma_2 \underset{1}{\dot{-}} \gamma_1)(d_1, d_4) = \mathscr{k}_1(d_1, 0, 0, d_4) = \bar{\mathscr{g}}(d_1, 0, 0, d_1d_4)$$

Let $\mathscr{k}_2$: $D^2 \vee D \to M$ be the function such that for any $(d_1, d_4, d) \in D^2 \vee D$,

$$(5.55) \quad \mathscr{k}_2(d_1, d_4, d) = \bar{\mathscr{g}}(d_1, 0, 0, d)$$

Then it follows from (5.54) and (5.55) that for any $(d_1, d_4) \in D^2$,

$$(5.56) \quad \tilde{t}_1(d_1, d_4) = \mathscr{k}_2(d_1, d_4, 0)$$
$$(5.57) \quad (\gamma_2 \underset{1}{\dot{-}} \gamma_1)(d_1, d_4) = \mathscr{k}_2(d_1, d_4, d_1d_4)$$

Therefore for any $d \in D$,

$$(5.58) \quad ((\gamma_2 \underset{1}{\dot{-}} \gamma_1) \dot{-} \tilde{t}_1)(D) = \mathscr{k}_2(0, 0, d) = \bar{\mathscr{g}}(0, 0, 0, d) = (\gamma_2 \dot{-} \gamma_1)(d) \quad \blacksquare$$

Proposition 5.8. Let $\alpha \in \mathrm{T}^2(M; m)$ with $K(\alpha) = (t_1, t_2)$. Then

$$(5.59) \quad \bar{\alpha}_3 \underset{1}{\dot{-}} \bar{t}_{13} = \Sigma(\tilde{t}_2)$$
$$(5.60) \quad \bar{\alpha}_3 \underset{2}{\dot{-}} \bar{t}_{23} = \Sigma(\tilde{t}_1)$$

where for any $(d_1, d_2, d_3) \in D^3$ and any $(e_1, e_2) \in D^2$,

$$(5.61) \quad \bar{\alpha}_3(d_1, d_2, d_3) = \alpha(d_1d_3, d_2d_3)$$
$$(5.62) \quad \bar{t}_{13}(d_1, d_2, d_3) = t_1(d_1d_3)$$
$$(5.63) \quad \bar{t}_{23}(d_1, d_2, d_3) = t_2(d_2d_3)$$

(5.64) $\tilde{t}_1(e_1, e_2) = \alpha(e_1, 0)$
(5.65) $\tilde{t}_2(e_1, e_2) = \alpha(0, e_1)$

Proof. Here we deal only with (5.59), leaving a similar treatment of (5.60) to the reader. Let $\not{f}$: $D^3 \underset{1}{\vee} D \to M$ be the function such that for any $(d_1, d_2, d_3, d_4) \in D^3 \underset{1}{\vee} D$,

(5.66) $\not{f}(d_1, d_2, d_3, d_4) = \alpha(d_1 d_3, d_4)$

Then it is easy to see that for any $(d_1, d_2, d_3) \in D^3$,

$$\begin{aligned}
(5.67)\quad \bar{t}_{13}(d_1, d_2, d_3) &= \alpha(d_1 d_3, 0) \\
&= \not{f}(d_1, d_2, d_3, 0) \\
(5.68)\quad \bar{\alpha}_3(d_1, d_2, d_3) &= \alpha(d_1 d_3, d_2 d_3) \\
&= \not{f}(d_1, d_2, d_3, d_2 d_3)
\end{aligned}$$

Therefore for any $(d_1, d_4) \in D^2$,

$$\begin{aligned}
(5.69)\quad (\bar{\alpha}_3 \underset{1}{\dot{-}} \bar{t}_{13})(d_1, d_4) &= \not{f}(d_1, 0, 0, d_4) \\
&= \alpha(0, d_4) \\
&= (\Sigma(\tilde{t}_2))(d_4) \quad \blacksquare
\end{aligned}$$

The rest of this section is devoted to simplicial object $D[3; 1, 2]$. We now define a kind of strong difference in $\mathsf{S}_{D[3;1,2]}(M; m)$.

Lemma 5.9. The diagram

$$\begin{tikzcd}
D(3) \arrow[r, "i"] \arrow[d, "i"'] & D[3;1,2] \arrow[d, "\nu"] \\
D[3;1,2] \arrow[r, "\mu"'] & D[3;1,2] \vee D(2)
\end{tikzcd}$$

is a quasi-colimit diagram of small objects, where

(5.70) $\mu(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0)$ for any $(d_1, d_2, d_3) \in D[3; 1, 2]$;
(5.71) $\nu(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1 d_3, d_2 d_3)$ for any $(d_1, d_2, d_3) \in D[3; 1, 2]$;
(5.72) $D[3; 1, 2] \vee D(2) = D^5(\{(1, 2), (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\})$.

Proposition 5.10. For any $\gamma_1, \gamma_2 \in \mathsf{S}_{D[3;1,2]}(M, m)$ with $\gamma_1|_{D(3)} = \gamma_2|_{D(3)}$, there exists a unique function τ: $D[3; 1, 2] \vee D(2) \to M$ such that $\tau \circ \mu = \gamma_1$ and $\tau \circ \nu = \gamma_2$. In this case we define a function $\gamma_2 \dot{-} \gamma_1$: $D(2) \to M$ such that for any $(d_1, d_2) \in D(2)$,

(5.73) $(\gamma_2 \dot{-} \gamma_1)(d_1, d_2) = \tau(0, 0, 0, d_1, d_2)$

Proof. Follows from Lemma 5.9. ■

Given $\tau \in \mathsf{S}_{D(3)}(M;\ m)$, we denote by $\mathsf{S}_{D[3;1,2]}(M;\ \tau)$ the subset of $\mathsf{S}_{D[3;1,2]}(M;\ m)$ consisting of all $\gamma \in \mathsf{S}_{D[3;1,2]}(M;\ m)$ with $\gamma|_{D(3)} = \tau$. We could make $\mathsf{S}_{D[3;1,2]}(M;\ \tau)$ an affine space over $\mathsf{T}^1(M;\ m) \times \mathsf{T}^1(M;\ m)$ with respect to the above strong difference and an appropriately defined strong translation, the details of which are safely left to the reader.

By the same token as Proposition 5.7, we have the following version of Proposition 7 of Lavendhomme (1996, §3.4) for $D[3;\ 1,\ 2]$.

Proposition 5.11. Let $\gamma_1, \gamma_2 \in \mathsf{S}_{D[3;1,2]}(M, m)$ with $\gamma_1|_{D(3)} = \gamma_2|_{D(3)}$. Let $t_1, t_2 \in \mathsf{T}^1(M;\ m)$ be such that $t_1 \oplus t_2 = \gamma_1 \dot{-} \gamma_2$. Then

$$(5.74)\quad ((\gamma_2 \underset{3}{-} \gamma_1) \underset{1}{-} \tilde{t}_1) \underset{2}{-} \tilde{t}_2 = \bar{t}_1 + \bar{t}_2$$

where

(5.75) $\tilde{t}_1(d_1, d_2, d_3) = \gamma_1(d_1, 0, 0) = \gamma_2(d_1, 0, 0)$ for any $(d_1, d_2, d_3) \in D[3;\ 1,\ 3]$;
(5.76) $\tilde{t}_1(d_1, d_2, d_3) = \gamma_1(0, d_2, 0) = \gamma_2(0, d_2, 0)$ for any $(d_1, d_2, d_3) \in D[3;\ 1,\ 3]$;
(5.77) $\bar{t}_1(d_1, d_3) = t_1(d_1 d_3)$ for any $(d_1, d_3) \in D^2$;
(5.78) $\bar{t}_2(d_2, d_3) = t_2(d_2 d_3)$ for any $(d_2, d_3) \in D^2$;
(5.79) $\bar{t}_1 \oplus \bar{t}_2 = \Sigma_{(123)}(\Sigma(\bar{t}_1) \oplus \Sigma(\bar{t}_2))$.

6. CURVATURE

The following proposition on the curvature form Ω is a quotation from Lavendhomme (1996, §5.3, Proposition 7) in our own terms.

Proposition 6.1. Let $\gamma \in \mathsf{T}^3(M;\ m)$, Then

$$(6.1)\quad \Omega(\gamma) = ((\gamma \underset{1}{\dot{-}} \tilde{\nabla}_{23}\gamma) \dot{-} \tilde{\nabla}(\gamma \underset{1}{\dot{-}} \tilde{\nabla}_{23}\gamma)) - ((\gamma \underset{2}{\dot{-}} \tilde{\nabla}_{13}\gamma) \dot{-} \tilde{\nabla}(\gamma \underset{2}{\dot{-}} \tilde{\nabla}_{13}\gamma))$$

Theorem 6.2. The simplicial form Ω is essentially tensorial.

Proof. By Corollary 4.8 the D^3-form Ω is essentially a $D[3;\ 1,\ 2]$-form. Therefore it suffices to show that for any $\gamma_1, \gamma_2 \in \mathsf{S}_{D[3;1,2]}(M;\ m)$ with $\gamma_1|_{D(3)} = \gamma_2|_{D(3)}$, we have $\Omega(\gamma_1) = \Omega(\gamma_2)$. Let $t_1, t_2 \in \mathsf{T}^1(M;\ m)$ such that $t_1 \oplus t_2 = \gamma_2 \dot{-} \gamma_1$. Let $\alpha = \nabla(t_1, t_2)$. Let $\bar{\alpha}_3$ be as in (5.61). Since

$$\begin{aligned}(6.2)\quad \bar{\alpha}_3(d_1, d_2, 0) &= \alpha(0, 0)\\ &= m\\ (6.3)\quad \bar{\alpha}_3(d_1, 0, d_3) &= \alpha(d_1 d_3, 0)\\ &= t_1(d_1 d_3)\end{aligned}$$

for any $(d_1, d_2, d_3) \in D^3$, we have $\tilde{\nabla}_{23}\overline{\alpha}_3 = \overline{t}_{13}$, where $\overline{t}_{13}$ is as in (5.62). Therefore $\overline{\alpha}_3 \underset{1}{\dot{-}} \tilde{\nabla}_{23}\overline{\alpha}_3 = \Sigma(\tilde{t}_2)$ by Proposition 5.8, where $\tilde{t}_2$ is as in (5.65). This means that

$$(6.4) \quad (\overline{\alpha}_3 \underset{1}{\dot{-}} \tilde{\nabla}_{23}\overline{\alpha}_3) \dot{-} \tilde{\nabla}(\overline{\alpha}_3 \underset{1}{\dot{-}} \nabla_{23}\overline{\alpha}_3) = 0$$

By the same token, we have

$$(6.5) \quad (\overline{\alpha}_3 \underset{2}{\dot{-}} \tilde{\nabla}_{13}\overline{\alpha}_3) \dot{-} \tilde{\nabla}(\overline{\alpha}_3 \underset{2}{\dot{-}} \tilde{\nabla}_{13}\overline{\alpha}_3) = 0$$

It follows from (6.4) and (6.5) that

$$(6.6) \quad \Omega(\overline{\alpha}_3) = 0$$

On the other hand, by using the notation of Proposition 5.11, we have $\overline{\alpha}_3 |_{D[3;1,2]} = (\overline{t}_1 \oplus \overline{t}_2)|_{D[3;1,2]}$. Therefore it follows from (6.6) and (5.74) that

$$(6.7) \quad \Omega(((\gamma_2 \underset{3}{\dot{-}} \gamma_1) \underset{1}{\dot{-}} \tilde{t}_1) \underset{2}{\dot{-}} \tilde{t}_2) = 0$$

Since $\Omega(\tilde{t}_1) = \Omega(\tilde{t}_2) = 0$, we have $\Omega(\gamma_2) = \Omega(\gamma_1)$, which completes the proof. ■

Theorem 6.3. Let $\gamma \in \mathrm{T}^3(M; m)$ with $K(\gamma) = (t_1, t_2, t_3)$. Then we have

$$(6.8) \quad \Omega(\gamma) = \Sigma_{(12)}(\nabla(\Sigma(\nabla(t_1, t_2)), \nabla(t_2, t_3))) \dot{-} \nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))$$

Proof. Since $\gamma|_{D(3)} = \nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))|_{D(3)}$ and the simplicial form Ω is tensorial by Theorem 6.2, we can take γ to be $\nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))$. Since $\gamma = \tilde{\nabla}_{23}\gamma$, we have

$$(6.9) \quad (\gamma \underset{1}{\dot{-}} \nabla_{23}\gamma) \dot{-} \nabla(\gamma \underset{1}{\dot{-}} \nabla_{23}\gamma) = 0$$

On the other hand, since $\tilde{\nabla}_{13}\gamma = \Sigma_{(12)}(\nabla(\Sigma(\nabla(t_1, t_2)), \nabla(t_2, t_3)))$, we have $\gamma|_{D[3]} = \tilde{\nabla}_{13}\gamma|_{D[3]}$, which implies, by dint of Proposition 5.7, that

$$(6.10) \quad (\gamma \underset{2}{\dot{-}} \tilde{\nabla}_{13}\gamma) \dot{-} \tilde{\nabla}(\gamma \underset{2}{\dot{-}} \tilde{\nabla}_{13}\gamma) = \gamma \dot{-} \tilde{\nabla}_{13}\gamma$$

The desired conclusion follows from (6.9) and (6.10). ■

The following theorem is a version of the celebrated Bianchi's first identity under the assumption that the connection ∇ is torsion-free.

Theorem 6.4. Let $\gamma \in \mathrm{T}^3(M, m)$ and $\rho = (132)$. The connection ∇ is assumed to be symmetric. Then we have

$$(6.11) \quad \Omega(\gamma) + \Omega(\Sigma_\rho(\gamma)) + \Omega(\Sigma_{\rho^2}(\gamma)) = 0$$

Proof. Since the connection ∇ is symmetric, it follows from Theorem 6.3 that

$$(6.12) \quad \Omega(\gamma) = \Sigma_{(12)}(\nabla(\Sigma(\nabla(t_1, t_2)), \nabla(t_2, t_3))) \\ \dot{-} \nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))$$

$$= \Sigma_{(12)}(\nabla(\nabla(t_2, t_1), \nabla(t_2, t_3))) \\ \dot{-} \nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))$$

By replacing γ by $\Sigma_\rho(\gamma)$ in (6.12) and noting that $(123)(12) = (13)$, $(123)(23) = (12)$, and $(13)(23) = (132)$, we have

$$\text{(6.13)} \quad \Omega(\Sigma_\rho(\gamma)) = \Sigma_{(12)}(\nabla(\nabla(t_3, t_2), \nabla(t_3, t_1))) \\ \dot{-} \nabla(\nabla(t_2, t_3), \nabla(t_2, t_1)) \\ = \Sigma_{(123)}(\Sigma_{(12)}(\nabla(\nabla(t_3, t_2), \nabla(t_3, t_1)))) \\ \dot{-} \Sigma_{(123)}(\nabla(\nabla(t_2, t_3), \nabla(t_2, t_1))) \\ = \Sigma_{(13)}(\nabla(\nabla(t_3, t_2), \nabla(t_3, t_1))) \\ \dot{-} \Sigma_{(123)}(\nabla(\nabla(t_2, t_3), \nabla(t_2, t_1))) \\ = \Sigma_{(13)}(\Sigma_{(23)}(\nabla(\nabla(t_3, t_1), \nabla(t_3, t_2)))) \\ \dot{-} \Sigma_{(123)}(\Sigma_{(23)}(\nabla(\nabla(t_2, t_1), \nabla(t_2, t_3))) \\ = \Sigma_{(132)}(\nabla(\nabla(t_3, t_1), \nabla(t_3, t_2))) \\ \dot{-} \Sigma_{(12)}(\nabla(\nabla(t_2, t_1), \nabla(t_2, t_3))$$

By replacing γ by $\Sigma_{\rho^2}(\gamma)$ in (6.12) and noting that $(12)(23) = (123)$ and $(123)^{-1} = (132)$, we have

$$\text{(6.14)} \quad \Omega(\Sigma_{\rho^2}(\gamma)) = \Sigma_{(12)}(\nabla(\nabla(t_1, t_3), \nabla(t_1, t_2))) \\ \dot{-} \nabla(\nabla(t_3, t_1), \nabla(t_3, t_2)) \\ = \Sigma_{(12)}(\Sigma_{(23)}(\nabla(\nabla(t_1, t_2), \nabla(t_1, t_3)))) \\ \dot{-} \nabla(\nabla(t_3, t_1), \nabla(t_3, t_2)) \\ = \Sigma_{(123)}(\nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))) \\ \dot{-} \nabla(\nabla(t_3, t_1), \nabla(t_3, t_2)) \\ = \Sigma_{(132)}(\Sigma_{(123)}(\nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))) \\ \dot{-} \Sigma_{(132)}(\nabla(\nabla(t_3, t_1), \nabla(t_3, t_2))) \\ = \nabla(\nabla(t_1, t_2), \nabla(t_1, t_3)) \\ \dot{-} \Sigma_{(132)}(\nabla(\nabla(t_3, t_1), \nabla(t_3, t_2)))$$

Letting $\gamma_1 = \Sigma_{(12)}(\nabla(\nabla(t_2, t_1), \nabla(t_2, t_3)))$, $\gamma_2 = \nabla(\nabla(t_1, t_2), \nabla(t_1, t_3))$, and $\gamma_3 = \Sigma_{(132)}(\nabla(\nabla(t_3, t_1), \nabla(t_3, t_2)))$ in Theorem 5.4, we obtain the desired identity from (6.12)–(6.14). ■

We recall that, given $X, Y, Z \in \chi^1(M)$, $R(X, Y)Z$ is defined as follows:

$$\text{(6.15)} \quad (R(X, Y)Z)_m = \Omega((Z * Y * X)_m$$

Corollary 6.5. We continue to assume that the connection ∇ is symmetric. For any $X, Y, Z \in \chi^1(M)$, we have the following identity:

$$\text{(6.16)} \quad R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

Proof. Let ρ be the same as in the above theorem. Then it is easy to see that

$$(6.17) \quad (X * Z * Y)_m|_{D(3)} = \Sigma_\rho((Z * Y * X)_m)|_{D(3)}$$
$$(6.18) \quad (Y * X * Z)_m|_{D(3)} = \Sigma_{\rho^2}((Z * Y * X)_m)|_{D(3)}$$

Therefore the desired result follows from Theorems 6.2 and 6.4. ■

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